ISRAEL JOURNAL OF MATHEMATICS 127 (2002), 317–357

RAMSEY–MILMAN PHENOMENON, URYSOHN METRIC SPACES, AND EXTREMELY AMENABLE GROUPS

BY

VLADIMIR PESTOV

School of Mathematical and Computing Sciences, Victoria University of Wellington
P.O. Box 600, Wellington, New Zealand
e-mail: vova@mcs.vuw.ac.nz
URL: http://www.mcs.vuw.ac.nz/~vova

ABSTRACT

In this paper we further study links between concentration of measure in topological transformation groups, existence of fixed points, and Ramsey-type theorems for metric spaces. We prove that whenever the group Iso(U) of isometries of Urysohn's universal complete separable metric space $\mathbb{U}$, equipped with the compact-open topology, acts upon an arbitrary compact space, it has a fixed point. The same is true if $\mathbb{U}$ is replaced with any generalized Urysohn metric space U that is sufficiently homogeneous. Modulo a recent theorem by Uspenskij that every topological group embeds into a topological group of the form Iso(U), our result implies that every topological group embeds into an extremely amenable group (one admitting an invariant multiplicative mean on bounded right uniformly continuous functions). By way of the proof, we show that every topological group is approximated by finite groups in a certain weak sense. Our technique also results in a new proof of the extreme amenability (fixed point on compacta property) for infinite orthogonal groups. Going in the opposite direction, we deduce some Ramsey-type theorems for metric subspaces of Hilbert spaces and for spherical metric spaces from existing results on extreme amenability of infinite unitary groups and groups of isometries of Hilbert spaces.

Received May 11, 2000 and in revised form December 1, 2000

1. Introduction

The concept of amenability extends from locally compact groups to arbitrary topological groups, and an interesting observation of recent times is that under such a transition the concept 'gains in strength' in that a number of concrete infinite-dimensional groups of importance satisfy a reinforced version of amenability such as locally compact groups cannot possibly have.

Definitions of amenability equivalent in the locally compact case diverge already for some of the most common infinite-dimensional topological groups [22]. Nevertheless, the following choice has become standard [35, 1]: call a topological group G **amenable** if every continuous affine action of G on a convex compact set has a fixed point. Equivalently, there is a left invariant mean on the space $C^b_r(G)$ of all bounded right uniformly continuous functions on G. This concept is in particular given substance by the following result due to de la Harpe [23]: a von Neumann algebra A is injective if and only if the unitary group $U(A)$ equipped with the ultraweak topology is amenable. (Cf. also [36].) Such results suggest that namely the above definition and not, for example, the one calling for an invariant mean on *all* bounded continuous functions on G, is the 'proper' choice.

In particular, a topological group G is amenable if it has a fixed point in *every* compact space it acts upon. Such topological groups are said to have the **fixed point on compacta property** (**f.p.c.**) [12], or else called **extremely amenable**, in the spirit of [17] where the concept was applied to discrete semigroups. The condition is equivalent to the existence of a left invariant **multiplicative** mean on $C^b_r(G)$.

At the first sight, the latter property seems to be far too restrictive to be observed *en masse*. In particular, according to a well-known theorem of Veech [48], no locally compact group has the fixed point on compacta property. (For discrete groups, this was previously noted in [7].) Historically the first examples of extremely amenable groups [26, 2], difficult to construct, looked like genuine pathologies.

Nevertheless, in recent times it was shown that a number of well-known 'massive' topological groups possess the fixed point on compacta property, among them

- the unitary group $U(\mathcal{H})$ (and the orthogonal group $\mathrm{O}(\mathcal{H})$) of an infinite-dimensional Hilbert space with the strong operator topology (Gromov and Milman [20]),
- the group $L_1(X, U(1))$ of measurable maps from a non-atomic Lebesgue

space to the circle group, equipped with the L_1-metric (Glasner [12] and independently, unpublished, Furstenberg and B. Weiss),

- groups $\mathrm{Homeo}_+(\mathbb{I})$ and $\mathrm{Homeo}_+(\mathbb{R})$ of orientation-preserving homeomorphisms with the compact-open topology (the present author [37]),
- groups of measure-preserving automorphisms of standard sigma-finite measure spaces with the strong topology (Giordano and the present author [11]).

The technique used to establish the fixed point on compacta property in the above examples has been either that of concentration of measure on high-dimensional structures (pioneered in this context by Gromov and Milman [20]), or else infinite Ramsey theory, as in [37].

In this article we isolate a new and vast class of topological groups with the fixed point on compacta property: they are groups of isometries of very regular and highly homogeneous objects, the (generalized) Urysohn metric spaces.

Universal metric spaces were introduced by Urysohn in the 20's [43, 44] and investigated mostly in the separable case. In particular, there is, up to an isometry, only one complete separable Urysohn metric space, which we will denote by $\mathbb{U}$. For a long time Urysohn spaces remained little known outside of general topology, and the most important advances at that period were due to Katětov [27], who had made the structure of the space $\mathbb{U}$ more transparent, and Uspenskij [45], who had proved that the group of isometries $\mathrm{Iso}(\mathbb{U})$ with the compact-open topology forms a universal second-countable topological group. Uspenskij's construction was later used by Gao and Kechris [10] to deduce, among others, the following result: every Polish topological group is the group of *all* isometries of a suitable separable complete metric space. Recently the Urysohn spaces were linked to wider issues in geometry and analysis, particularly by Vershik who has for example shown [50] that the completion of the set of integers equipped with a 'sufficiently random' metric is almost surely isometric to $\mathbb{U}$. A further discussion of the space $\mathbb{U}$ and its links with geometry is to be found in Gromov's book [19].

We shall prove that the group $\mathrm{Iso}(\mathbb{U})$ has the fixed point on compacta property (Theorem 4.11), and moreover the same is true of isometry groups $\mathrm{Iso}(U)$ of all sufficiently homogeneous generalized (non-separable) Urysohn spaces U (Theorem 6.6). According to a recent result by Uspenskij [47], every topological group is contained, as a subgroup, in the group of isometries of such a generalized Urysohn space. The two results combined imply that extreme amenability is, in a sense, ubiquitous: every topological group embeds, as a topological subgroup, into a topological group with the fixed point on compacta property (Corollary 6.7).

It is known since the work of de la Harpe [22] that a closed subgroup of an amenable topological group need not be amenable, unlike in the locally compact case. The reported results take this observation to its extreme. The possibility of such a development was conjectured in our paper [37].

The proof of extreme amenability of the group $\mathrm{Iso}(\mathbb{U})$ applies the technique of concentration of measure, and by way of proof we establish the following generalization of a result due to Glasner and Furstenberg–Weiss: the group of all measurable maps from a non-atomic Lebesgue measure space to an amenable locally compact group G, equipped with the topology of convergence in measure (known as the Hartman–Mycielski extension of G, [25]), has the fixed point on compacta property (Theorem 2.2). Another component of the proof is the following, apparently new, result (Theorem 3.2): every group of isometries of a metric space can be approximated in a certain weak sense with finite groups of isometries of suitable metric spaces. In the second-countable case the result can be interpreted as a statement on approximation of *topological* groups: every Polish group is the limit of a net of finite groups in the space of all closed subgroups of the group $\mathrm{Iso}(\mathbb{U})$ (Corollary 4.9).

Our methods lead to a new proof of the fixed point on compacta property for the infinite orthogonal groups with the strong topology, which does not use advanced geometric tools such as Gromov's isoperimetric inequality. (Subsection 4.5.)

In order to extend the result on extreme amenability to the groups of isometries $\mathrm{Iso}(U)$ of generalized Urysohn metric spaces U, we recast the fixed point on compacta property of the full isometry group of a sufficiently homogeneous metric space X as a Ramsey-type result for the space X itself (Theorem 5.9). As a corollary, if two metric spaces, X and Y, are both ω-homogeneous and have, up to isometry, the same finite metric subspaces, then the groups $\mathrm{Iso}(X)$ and $\mathrm{Iso}(Y)$ have the fixed point on compacta property (or otherwise) simultaneously (Theorem 6.5).

As another application of this technique, we show that the groups of isometries of the universal discrete metric spaces [19] do not have the fixed point on compacta property (Theorem 6.9).

The equivalence between the fixed point on compacta property of isometry groups and Ramsey-type results for metric spaces can be exploited in the other direction as well, and thus we deduce some 'approximate' Ramsey-type results for both spherical and Euclidean metric spaces (Subsection 6.3).

ACKNOWLEDGEMENT: The investigation grew out of stimulating discussions

with Mikhaïl Gromov and Vladimir Uspenskij, who have both independently conjectured extreme amenability of the isometry groups of Urysohn spaces. Moreover, Theorems 3.2 and 5.7 are rigorous incarnations of two further conjectures made by VU. My thanks go to both named mathematicians, to MG also for hospitality extended at IHÉS in September 1999. I am grateful to Alekos Kechris and to Eli Glasner for spotting badly flawed fragments in the earlier versions of the article, and to Michael Megrelishvili for a useful remark. I am thankful to Sina Greenwood for the invitation to give the opening talk at the 2000 Devonport Topology Festival, which had rekindled my work on the topic, and to Michael Cowling for his hospitality at the Department of Pure Mathematics of the University of New South Wales in April 2000, where part of the work had been done. The research was partially supported by the Royal Society of New Zealand through the Marsden Fund grant VUW703, by the Australian Research Council through the large research grant 'Random algebraic structures,' and by the Institut des Hautes Études Scientifiques through its visitors programme.

2. Concentration of measure in Hartman–Mycielski groups

2.1. Our starting point is the following result, mentioned in the Introduction.

THEOREM 2.1 (Glasner [12]; Furstenberg–B. Weiss, unpublished): *The group $L_1(X, \mathrm{U}(1))$ of all measurable maps from a nonatomic Lebesgue space to the circle rotation group, equipped with the L_1-metric, has the fixed point on compacta property.*

On two occasions in this article, including the proof of one of our main theorems, we will invoke suitable modifications of the above result, and it seems appropriate to state a far-reaching generalization of Theorem 2.1, even if we shall never use its full power.

In the above form the result does not extend too far: suffice to consider the additive group of the Banach space $L_1(X) = L_1(X, \mathbb{R})$, with its wealth of continuous characters. However, it is not a particular metric on the group but rather the topology it generates that matters, and the topology generated by the L_1-metric on the group $L(X, \mathbb{T})$ is that of convergence in measure. (This is true of every L_p-metric, $1 \leq p < \infty$, on the same group.) This observation leads us to state the following generalization of Glasner–Furstenberg–Weiss theorem.

THEOREM 2.2: *Let G be an amenable locally compact group and let X be a non-atomic Lebesgue measure space. Then the group $L_0(X, G)$ of all measurable*

maps from X to G, equipped with the topology of convergence in measure, has the fixed point on compacta property (is extremely amenable).

Remark 2.3: The topological groups of the form $L_0(X, G)$, where the subscript '0' stands for the topology of convergence in measure, had apparently been first considered by Hartman and Mycielski [25], who had observed that $L_0(X, G)$ contains G as a topological subgroup (formed by all constant functions) and is path-connected and locally path–connected. Later it was shown by Keesling [28] that if G is separable metrizable, then the Hartman–Mycielski extension $L_0(X, G)$ is homeomorphic to the separable Hilbert space. The correspondence $G \mapsto L_0(X, G)$ determines a (covariant) functor from the category of all topological groups and continuous homomorphisms to itself, and Theorem 2.2 says that the Hartman–Mycielski functor transforms amenable locally compact groups into extremely amenable topological groups.

The following particular case (where $G = \mathbb{R}$ or $\mathbb{C}$) seems to be of interest.

COROLLARY 2.4: *The [underlying topological group of] the topological vector space $L_0(X)$ of all measurable functions on a non-atomic Lebesgue measure space X, equipped with the topology of convergence in measure, has the fixed point on compacta property.*

Remark 2.5: The above result is similar to the one from [26] where the space $L_0(X)$ was equipped with the topology of convergence in a suitably chosen, the so-called pathological submeasure (a subadditive set function) on X. As a result, the abelian topological group from [26] has an even stronger property than just extreme amenability: it admits no strongly continuous unitary representations. Notice that each of the groups of the form $L_0(X, G)$ from Theorem 2.2 admits a faithful strongly continuous unitary representation in the Hilbert space $L_2(X, L_2(G))$. (This extends an observation made in [12] for $G = U(1)$.)

Our proof of Theorem 2.2 relies, similarly to that of Theorem 2.1, on the technique of concentration of measure on high-dimensional structures. However, the concept of a Lévy group [20, 12] becomes too narrow and has to be somewhat extended. We believe that this extension goes sufficiently far to be of interest on its own. (Though we find it useful, to replace metrics with uniform structures, this is not what our generalization is about.)

2.2. If $X = (X, \mathcal{U}_X)$ is a uniform space, then the uniform (induced) topology on X gives rise to a Borel structure and thus one can speak of Borel measures on

X. The following is a straightforward adaptation of the by now classical concept [20, 34, 33, 41, 19, 12].

Definition 2.6: Let (μ_α) be a net of probability measures on a uniform space $(X, \mathcal{U}_X)$. Say that the net (μ_α) has the **Lévy concentration property**, or simply **concentrates** (in X), if whenever $A_\alpha \subseteq X$ are Borel subsets with the property

$$\liminf_\alpha \mu_\alpha(A_\alpha) > 0,$$

one has for every entourage of the diagonal $V \in \mathcal{U}_X$

$$\mu_\alpha(V[A_\alpha]) \to 1.$$

(Here, as usual, $V[A] = \{x \in X \mid \exists a \in A,\ (x,a) \in V\}$ denotes the V-neighbourhood of A.)

LEMMA 2.7: *Let $f\colon X \to Y$ be a uniformly continuous map between two uniform spaces, and let (μ_α) be a net of Borel measures on X. If (μ_α) concentrates, then the net $(f_*(\mu_\alpha))$ of push-forward measures on Y concentrates as well.*

Let G be a group of uniform isomorphisms of a uniform space X. A compactification K of X is called **uniform** if the corresponding mapping $i\colon X \to K$ is uniformly continuous, and **equivariant** (in full, **G-equivariant**) if G acts on K by homeomorphisms in such a way that i commutes with the action. The maximal uniform compactification of a uniform space X, known as the **Samuel compactification** of X and which we denote by σX, is the Gelfand space of the commutative C^*-algebra formed by all bounded uniformly continuous complex-valued functions on X. The Samuel compactification σX is equivariant no matter what the acting group G is, because every uniform homeomorphism $X \to X$ extends to a self-homeomorphism $\sigma X \to \sigma X$ due to universality.

It is convenient to state explicitely the following result, which is in essence folk's knowledge in theory of topological transformation groups. (Cf. [33, 48] etc.)

THEOREM 2.8: *Let G be a group of uniform isomorphisms of a uniform space X. The following conditions are equivalent.*

(i) *Every G-equivariant uniform compactification of X has a fixed point.*

(ii) *For every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$ and every finite collection $g_1, g_2, \ldots, g_n \in G$, there is an $x \in X$ with $|f(x) - f(g_i x)| < \varepsilon$ for all $i = 1, 2, \ldots, n$.*

(iii) *For every finite cover γ of X, every $V \in \mathcal{U}_X$ and every finite collection $g_1, g_2, \ldots, g_n \in G$, there is an $A \in \gamma$ such that*

$$\bigcap_{i=1}^{n} V[g_i A] \neq \emptyset.$$

Proof: (i) $\Rightarrow$ (ii): notice that every bounded uniformly continuous function extends over the Samuel compactification.

(ii) $\Rightarrow$ (iii): choose a bounded uniformly continuous pseudometric d on X subordinated to V (that is, $d(x,y) < 1 \Rightarrow (x,y) \in V$) and apply the condition (ii) to the function from X to $\mathbb{R}^{|\gamma|}$ whose components are distance functions $x \mapsto d(x, A)$, $A \in \gamma$, with $\varepsilon = 1$.

(iii) $\Rightarrow$ (i): see [39], Proposition 2.1 (which is, in its turn, an adaptation of an argument from Section 4 in [32]). ∎

Remark 2.9: At this point we do not concern ourselves with a topology on the acting group G, and it may well happen that if G is a topological group acting on the uniform space X continuously, the extension of the action to an equivariant compactification of X is discontinuous. As an example, consider as G the unitary group $U(l_2)$ with the strong topology, and as X the unit sphere $\mathbb{S}^\infty$ in l_2 with the metric uniformity. The action of $U(l_2)$ on the Samuel compactification of the sphere is continuous if $U(l_2)$ is equipped with the uniform operator topology, but not the strong one.

A subset B of a uniform space X is called **uniformly open** if $B = V[A]$ for some $A \subseteq X$ and $V \in \mathcal{U}_X$.

Definition 2.10: Let us say that two nets of probability measures, (μ_α) and (ν_α), on the same uniform space X are **asymptotically proximal** if for every uniformly open subset B one has

$$\limsup_\alpha |\mu_\alpha(B) - \nu_\alpha(B)| < 1.$$

Remark 2.11: Two nets as above will in particular be asymptotically proximal if $\liminf_\alpha (\mu_\alpha \wedge \nu_\alpha)(X) > 0$. For instance, this is so if the restrictions of μ_α and ν_α coincide on some Borel subsets (A_α), whose measures are uniformly (in α) bounded away from zero.

The following apparently subsumes all the previously known results of the type (*concentration of measure*) $\Rightarrow$ (*existence of a fixed point*) [20, 32, 33, 12, 30].

THEOREM 2.12: *Let a group G act on a uniform space $X = (X, \mathcal{U})$ by uniform isomorphisms. Suppose there is a net (μ_α) of probability measures on X such that*

- *(μ_α) concentrates in X,*
- *for every $g \in G$ the nets (μ_α) and $(g * \mu_\alpha)$ are asymptotically proximal.*

Then every equivariant uniform compactification of the G-space X has a fixed point.

Remark 2.13: The second condition is a rather weak invariance-type property for a family of measures, and its advantage is being easier to verify. If we require all measures μ_α to be **eventually invariant** (that is, for every $g \in G$ one has $\mu_\alpha = g * \mu_\alpha$ for sufficiently large α) and compactly-supported, then we recover the concept of a Lévy transformation group from [32]. The above stated theorem allows for a unified approach to a number of previously known results, such as a link between amenability of unitary representations and the concentration property of unit spheres [39], which we will not be addressing here.

Proof: Let γ be a finite cover of X, let $V \in \mathcal{U}_X$, and let $g_1, \ldots, g_n \in G$ be arbitrary. Find an entourage of the diagonal $W \in \mathcal{U}_X$ with $W \circ W \subseteq V$. At least one element of γ, denote it by A, satisfies the property

$$\limsup_\alpha \mu_\alpha(A) \geq |\gamma|^{-1}.$$

By proceeding to a subnet if necessary, we may assume without loss in generality that

$$\liminf_\alpha \mu_\alpha(A) \geq |\gamma|^{-1}.$$

In view of the assumed concentration property of the measures (μ_α),

$$\lim_\alpha \mu_\alpha(W[A]) = 1,$$

and by force of the second assumption, one has for every i

$$\liminf_\alpha (g_i * \mu_\alpha)(W[A]) > 0.$$

By Lemma 2.7, each of the nets of measures $(g_i * \mu_\alpha)$, $i = 1, 2, \ldots, n$ concentrates, and consequently

$$\lim_\alpha (g_i * \mu_\alpha)(W[W[A]]) = 1.$$

Since $W \circ W \subseteq V$, one has

$$\lim_\alpha \mu_\alpha(g_i V[A]) = 1.$$

It is therefore possible to choose an α so large that each of the numbers $\mu_\alpha(g_1V[A]), \ldots, \mu_\alpha(g_nV[A])$ is greater than $1 - 1/n$. It follows that the intersection of all the translates of $V[A]$ by elements g_i, $i = 1, 2, \ldots, n$ is non-empty, and application of Theorem 2.8 finishes the proof. ∎

2.3. Recall that the **right uniform structure** of a topological group, $\mathcal{U}_r(G)$, has as a basis the entourages of diagonal of the form

$$V_r = \{(x, y) \in G \times G \mid xy^{-1} \in V\},$$

where V runs over a neighbourhood basis of e in G. The Samuel compactification of the right uniform space $G_r = (G, \mathcal{U}_r(G))$ is a compact G-space, known as the **greatest ambit** of G and denoted by $\mathcal{S}(G)$. (Cf. [42, 5, 1, 38].) The greatest ambit possesses a distinguished point (the image of identity of G, which we will still denote e), whose orbit is everywhere dense in it. This object has the following universal property: whenever X is a compact G-space and $x_0 \in X$, there is a unique morphism of G-spaces from $\mathcal{S}(G)$ to X taking e to x_0. It follows that a topological group G has the fixed point on compacta property if and only if there is a fixed point in the greatest ambit $\mathcal{S}(G)$.

COROLLARY 2.14: *Let G be a topological group. Suppose there is a net (μ_α) of probability measures on G such that, with respect to the right uniform structure $\mathcal{U}_r(G)$,*

- *(μ_α) concentrates,*
- *for every $g \in G$ the nets (μ_α) and $(g * \mu_\alpha)$ are asymptotically proximal.*

Then G has the fixed point on compacta property.

Remark 2.15: A topological group G is called a **Lévy group** if it contains a family of compact subgroups, directed by inclusion and having everywhere dense union, such that the corresponding normalized Haar measures, μ_α, concentrate in G_r. This concept was used as means to deduce the existence of fixed points for group actions on compacta by Gromov and Milman [20]; see also [12, 38]. Lévy groups satisfy a stronger property than the the second assumption of Corollary 2.14: the measures μ_α are eventually invariant.

2.4. Let $X = (X, \mathcal{U}_X)$ be a uniform space. Denote by $L(\mathbb{I}, X)$ the collection of all Borel-measurable maps $f\colon \mathbb{I} \to X$ equipped with the uniform structure of convergence in measure. The standard basic entourages of diagonal are of the form

$$[V, \varepsilon] := \{(f, g) \in L(\mathbb{I}, X) \times L(\mathbb{I}, X)\colon \mu\{x \in \mathbb{I}\colon (f(x), g(x)) \notin V\} < \varepsilon\},$$

where $V \in \mathcal{U}_X$ and $\varepsilon > 0$. This uniformity induces a topology on $L(\mathbb{I}, X)$, whose standard basic neighbourhoods of a given function $f\colon \mathbb{I} \to X$ are

$$[V, \varepsilon, f] := \{g \in L(\mathbb{I}, X)\colon \mu\{x \in \mathbb{I}\colon (f(x), g(x)) \notin V\} < \varepsilon\},$$

where $V \in \mathcal{U}_X$ and $\varepsilon > 0$. (Notice that the knowledge of topology on X alone does not suffice: to talk of convergence in measure, it is necessary to have a uniform structure, for instance, one defined by a metric on X, or else the unique compatible uniform structure in case X is compact.)

If G is a Hausdorff topological group, then so is $L(\mathbb{I}, G)$. In this case, the standard neighbourhoods of identity are of the form

$$[V, \varepsilon] := \{g \in L(\mathbb{I}, X)\colon \mu\{x \in \mathbb{I}\colon g(x) \notin V\} < \varepsilon\},$$

where V is a neighbourhood of e_G in G and $\varepsilon > 0$.

Now suppose that $X = (X, \rho)$ is a metric space. Let us agree on the canonical choice of the metric generating the uniformity of convergence in measure (and the corresponding topology) on $L_0(\mathbb{I}, X)$, as follows: if $\lambda > 0$ is an arbitrary (but fixed) number, then set

$$\mathrm{me}_\lambda(f, g) = \inf\{\varepsilon > 0 \mid \mu\{x \in \mathbb{I}\colon \rho(f(x), g(x)) > \varepsilon\} < \lambda\varepsilon\}. \tag{2.1}$$

Such metrics for different $\lambda > 0$ are all equivalent. (Cf. [19], p. 115.)

Definition 2.16: An action of a topological group G on a uniform space $X = (X, \mathcal{U}_X)$ by uniform isomorphisms is called **bounded** [51] (or **motion equicontinuous** [14]) if for every entourage $U \in \mathcal{U}_X$ one can find a neighbourhood $W \ni e_G$ such that for every $x \in X$,

$$W \cdot x \subseteq U[x].$$

Remarks 2.17: 1. Every bounded action is continuous. [If $g \in G$, $x \in X$, and a neighbourhood $\mathcal{O} \ni g \cdot x$ are arbitrary, select an $U \in \mathcal{U}_X$ with $(U \circ U)[x] \subseteq \mathcal{O}$ and a neighbourhood $W \ni e_G$ with $W \cdot y \subseteq U[y]$ for all $y \in X$. Then $W \cdot U[x] \subseteq U^2[x] \subseteq \mathcal{O}$.]

2. The converse is not true. [For example, the standard action of the unitary group $U(l_2)$ with the *strong* operator topology on the unit sphere $\mathbb{S}^\infty$ equipped with the metric uniformity is continuous, but not bounded. This action becomes bounded if $U(l_2)$ is equipped with the *uniform* operator topology.]

3. However, a continuous action of a topological group G on a *compact* space X (equipped with the unique compatible uniformity) is always bounded. This fact is well-known (and easily verified).

4. The left action of a topological group G on the *right* uniform space $G_{\mathfrak{r}}$ is bounded, but in general the same is not true of the left action of G on the *left* uniform space of G.

LEMMA 2.18: *If a topological group G acts by isometries on a metric space X, then the topological group $L_0(\mathbb{I}, G)$ acts by isometries on the metric space $L(\mathbb{I}, X)$ equipped with the metric (2.1), where the action is defined pointwise:*

$$(g \cdot f)(x) := g(x) \cdot f(x),\ g \in L(\mathbb{I}, G),\ f \in L(\mathbb{I}, X).$$

If in addition the action of G on X is bounded (for example X is compact), then the action of $L_0(\mathbb{I}, G)$ on $L_0(\mathbb{I}, X)$ is continuous.

Proof: The first statement is self-evident. In order to establish the second claim, it is now enough to prove that for every $f \in L_0(\mathbb{I}, X)$ the orbit map

$$L_0(\mathbb{I}, G) \ni g \mapsto g \cdot f \in L_0(\mathbb{I}, X)$$

is continuous. Let $\varepsilon > 0$ be any. Using the boundedness of the original action, choose a $W \ni e_G$ such that for all $x \in X$ and $w \in W$, $\rho(w \cdot x, x) < \varepsilon$. The set $g[W, \lambda\varepsilon/2]$ is a neighbourhood of g in $L_0(\mathbb{I}, G)$, and if $g_1 \in g[W, \lambda\varepsilon/2]$ is arbitrary, then for every $x \in X$ apart from a set of measure $\leq \lambda\varepsilon$ one has $\rho(g(x) \cdot f(x), g_1(x) \cdot f(x)) = \rho(f(x), w(x) \cdot f(x)) < \varepsilon$, where $w(x) \equiv g(x)^{-1} g_1(x) \in W$ for every $x \in X$ apart from a set of measure $\lambda\varepsilon/2$. This means that $\mathrm{me}_\lambda(g_1 \cdot f, g \cdot f) \leq \varepsilon$, establishing the continuity of the orbit map. ∎

2.5. PROOF OF THEOREM 2.2. Fix a parametrization of the non-atomic Lebesgue measure space X, that is, a measure space isomorphism $\mathbb{I} \leftrightarrow X$. The required net of measures on $L(\mathbb{I}, G)$ will be indexed by the set of all pairs of the form (n, F), where $n \in \mathbb{N}_+$ and $F \subseteq G$ is a finite subset, directed as follows: $(n_1, F_1) \prec (n_2, F_2)$ iff $n_1 \leq n_2$ and $F_1 \subseteq F_2$. Fix a left-invariant Haar measure ν on G. For every n, F as above use the Følner condition and the assumed amenability of the locally compact group G to choose a compact subset $K = K_{n,F} \subseteq G$ with the property

$$\frac{\nu(gK \Delta K)}{\nu(K)} < \frac{1}{n}$$

for each $g \in F$. Now let K^n denote the set of all functions in $L_0(\mathbb{I}, G)$ taking values in K and constant on every interval of the form $[i/n, (i+1)/n)$, $i = 0, 1, \ldots, n-1$. Topologically, K^n can be identified with the n-th power of the

compact set K. Denote by $\nu_{n,F}$ the product measure $(\nu|_K)^n$ normalized to one and viewed as a probability measure on $L_0(\mathbb{I}, G)$ with support K^n. It remains to verify that the net of probability measures $(\nu_{n,F})$ on the topological group $L_0(\mathbb{I}, G)$ satisfies the two assumptions of Corollary 2.14.

(i) The net of measures $(\nu_{n,F})$ concentrates in $L_0(\mathbb{I}, G)$.

The following general and powerful result, due to Talagrand ([41], p. 76 and Prop. 2.1.1), extends the particular case of finite spaces belonging to Schechtman [40, 34]. Let $Y = (Y, \Sigma, \mu)$ denote a probability space. Then the product measures $\mu^{\otimes n}$, $n \in \mathbb{N}$, on Y^n concentrate, as $n \to \infty$, with respect to the [uniform structure generated by the] normalized Hamming distance on Y^n, given by

$$\rho(f, g) = \frac{1}{n}\sharp\{i \mid f_i \neq g_i\}.$$

Moreover, the (Gaussian) bounds for the rate of concentration are independent of a particular Y, cf. *loco citato*. In other words, there is a family of functions $\alpha_n\colon [0,1] \to [0, \frac{1}{2}]$ (of the form $\alpha_n(\varepsilon) = C_1 \exp(-C_2 \varepsilon n^2)$), independent of Y and μ and such that, whenever a measurable $A \subseteq Y^n$ has the property $\mu^{\otimes n}(A) \geq \frac{1}{2}$, one has for every $\varepsilon > 0$

$$\mu^{\otimes n}(A_\varepsilon) \geq 1 - \alpha_n(\varepsilon),$$

where $A_\varepsilon = \{y \in Y^n\colon \rho(y, A) < \varepsilon\}$.

In view of Lemma 2.7, it is therefore enough to show that the uniform structure induced on K^n by the Hamming-type distance ρ is finer than the restriction of the right uniform structure $\mathcal{U}_r(L(X, G))$ (which of course coincides with the unique compatible uniformity on K^n). Let V be a neighbourhood of unity in G and let $\varepsilon > 0$. Let $f, g \in K^n$ be arbitrary and such that $\rho(f, g) < \varepsilon$. Then clearly

$$\begin{aligned} \mu(\{x \in X \mid f(x)g(x)^{-1} \notin V\}) &\leq \mu(\{x \in X \mid f(x) \neq g(x)\}) \\ &= \frac{1}{n}|\{i \mid f_i \neq g_i\}| \\ &= \rho(f, g) < \varepsilon, \end{aligned} \tag{2.2}$$

that is, $(f, g) \in [V; \varepsilon]$, establishing the claim.

(ii) For every $g \in L_0(\mathbb{I}, G)$, the nets $(\nu_{n,F})$ and $(g * \nu_{n,F})$ are asymptotically proximal.

Let $g \in L_0(X, G)$. By approximating g with simple functions, one can assume without loss in generality that the set $F = \{g_1, \ldots, g_k\}$ of values of g is finite, and that for sufficiently large n, the function g is constant on each $[i/n, (i+1)/n)$. Since for every g_i one has $\nu(K_{n,F} \cap g_i \cdot K_{n,F}) > (1 - 1/n)\nu(K_{n,F})$, it follows that,

whenever $n \gg k$,

$$\nu_{n,F}(K^n_{n,F} \cap g_i \cdot K^n_{n,F}) > \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e}.$$

To finish the proof, notice that the restrictions of the measures $\nu_{n,F} = (\nu|_K)^n$ and $g * \nu_{n,F}$ to $K^k_{n,F} \cap g_i \cdot K^k_{n,F}$ coincide, and use Remark 2.11. ∎

3. Approximation by finite groups

3.1. The aim of this Section is to show that *every* (topological) group can be approximated, albeit in a very weak sense, by finite groups. By combining the approximation result with the extreme amenability of Hartman–Mycielski groups, we shall later deduce the fixed point on compacta property for the isometry group $\mathrm{Iso}(\mathbb{U})$ of the complete separable Urysohn metric space.

We will state the approximation result in a few equivalent forms. Let us say that a metric space X is **indexed** by a set I if there is a surjection $f_X\colon I \to X$. We will call the pair (X, f_X) an **indexed metric space.** Let us say that two metric spaces, X and Y, indexed with the same set I are **ε-isometric** if for every $i, j \in I$ the distances $d_X(f_X(i), f_X(j))$ and $d_Y(f_Y(i), f_Y(j))$ differ by at most ε.

LEMMA 3.1: *If metric spaces X and Y indexed by a set I are ε-isometric, then X and Y can be isometrically embedded into a metric space Z in such a way that for each $i \in I$, $d_Z(f_X(i), f_Y(i)) \leq \varepsilon$.*

Proof: Make the set-theoretic disjoint union $Z = X \cup Y$ into a weighted graph, by joining a pair (x, y) with an edge in any of the following cases:

- $x, y \in X$, with weight $\rho_X(x, y)$;
- $x, y \in Y$, with weight $\rho_Y(x, y)$;
- for some $i \in I$, $x = f_X(i)$ and $y = f_Y(i)$, with weight ε.

The weighted graph Z equipped with the path metric clearly contains X and Y as metric subspaces and satisfies the required property. ∎

THEOREM 3.2: *Let $g_1, \ldots, g_n$ be a finite family of isometries of a metric space X. Then for every $\varepsilon > 0$ and every finite collection $x_1, \ldots, x_m$ of elements of X there exist a finite metric space $\tilde{X}$, elements $\tilde{x}_1, \ldots, \tilde{x}_m$ of $\tilde{X}$, and isometries $\tilde{g}_1, \ldots, \tilde{g}_n$ of $\tilde{X}$ such that the indexed metric spaces $\{g_i \cdot x_j\colon i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ and $\{\tilde{g}_i \cdot \tilde{x}_j\colon i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ are ε-isometric.*

Proof: We will perform the proof in several simple steps.

1. By first rescaling the metric ρ_X on X and then replacing it with $\min\{\rho_X, 1\}$ if necessary, we can assume without loss in generality that the values of ρ_X are bounded by 1.

2. Without loss in generality, we may also assume that X supports the structure of an abelian group equipped with a bi-invariant metric, and g_i's are metric-preserving group automorphisms. For instance, one can replace X with the free abelian group $A(X)$ on X, and extend the metric from X to a maximal invariant metric on $A(X)$ bounded by 1 (the so-called **Graev metric,** cf. [15, 46]); then every isometry of X uniquely extends to an isometric automorphism of the metric group $A(X)$.

3. Let G denote a group of isometries of X generated by $g_1, \ldots, g_n$. The semidirect product group $G \ltimes A(X)$ is equipped with the bi-invariant metric ρ defined by

$$\rho((g,a),(g',a')) = \begin{cases} 1, & \text{if } g \neq g', \\ d(a,a'), & \text{otherwise.} \end{cases}$$

[The bi-invariance of ρ is established through a direct calculation using the multiplication rule in the semidirect product in question:

$$(g,a)(h,b) = (gh, a + g \cdot b).]$$

As usual, we will identify G with a subgroup of the semidirect product under the mapping $G \ni g \mapsto (g,0)$, and similarly $A(X)$ is identified with a normal subgroup of the semidirect product under the mapping $A(X) \ni x \mapsto (e_G, x)$. Under such conventions, the automorphism of $A(X)$ determined by each $g \in G$ is just g itself considered as an isometric isomorphism of $A(X)$:

$$\begin{aligned} \forall a \in A(X), \quad gag^{-1} &\equiv (g,0)(e,a)(g,0)^{-1} \\ &= (g, g \cdot a)(g^{-1}, 0) \\ &= (e, g \cdot a) \\ &\equiv g \cdot a. \end{aligned}$$

In particular, for every i, j one has

$$g_i x_j g_i^{-1} = g_i \cdot x_j.$$

4. Let F_{m+n} denote the free group on $m+n$ generators denoted by the symbols

$$g_1, g_2, \ldots, g_n, x_1, x_2, \ldots, x_m.$$

Denote by $\pi\colon F_{m+n} \to G \ltimes A(X)$ the homomorphism sending each generator g_i to the corresponding element of G and each generator x_j to the corresponding

element of $X \subset A(X)$. Pull the metric ρ back from $G \ltimes A(X)$ to F_{m+n} by letting

$$\rho'(x, y) = \rho(\pi(x), \pi(y)).$$

The pseudometric ρ' is bi-invariant on F_{m+n}, though need not be a metric. By force of the remark at the end of step 3, the indexed pseudometric spaces

$$\{g_i \cdot x_j \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \subseteq (X, \rho_X) \subset (A(X), \rho)$$

and

$$\{g_i x_j g_i^{-1} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\} \subset (F_{m+n}, \rho')$$

are isometric (and thus are both metric spaces).

5. By adding to ρ' an arbitrary bi-invariant metric on F_{m+n} normalized so as to only slightly change the values of distances between pairs of elements $g_i x_j g_i^{-1}$ (for instance, let us agree on the discrete metric taking its values in $\{0, \varepsilon/2\}$)), we can assume without loss in generality that ρ' is a bi-invariant *metric* on F_{m+n}, while the indexed metric spaces $\{g_i \cdot x_j \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ and $\{g_i x_j g_i^{-1} \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$ are $\varepsilon/2$-isometric.

6. Now replace the metric ρ' with the **maximal** among all bi-invariant metrics on F_{m+n} that coincide with ρ' on the set $F_{m+n}^{(3)}$ of all words of reduced length ≤ 3.

To prove the existence of such a metric, say d, denote by $\mathcal{M}$ the family of all bi-invariant metrics on F_{m+n} whose restriction to $F_{m+n}^{(3)}$ coincides with ρ'. Since $\mathcal{M} \ni \rho'$, the family $\mathcal{M}$ is non-empty. For any two elements $w, v \in F_{m+n}$ and an arbitrary $\varsigma \in \mathcal{M}$, the value $\varsigma(w, v)$ is bounded from above uniformly in ς by any sum of the form $\sum_k \rho'(a_k, b_k)$, where $w = \sum_k a_k$ and $v = \sum_k b_k$ are two representations having the same length and such that $a_k, b_k \in F_{m+n}^{(3)}$. Now set

$$d(v, w) = \sup_{\varsigma \in \mathcal{M}} \varsigma(v, w).$$

The supremum on the r.h.s. is finite, and has all the required properties.

7. Notice that for $w, v \in F_{m+n}$

$$d(w, v) = \inf \sum_k \rho'(a_k, b_k),$$

where the infimum is taken over all possible representations of the above sort $w = \sum_k a_k$, $v = \sum_k b_k$, having the same length and such that $a_k, b_k \in F_{m+n}^{(3)}$.

[*Proof:* the infimum on the r.h.s. is a bi-invariant pseudometric, which is greater than or equal to d, and whose restriction to $F_{m+n}^{(3)}$ coincides with the restriction

of ρ'. We conclude: this infimum is in fact a metric, and it must coincide with d.]

8. Denote by δ the smallest positive value of the distance ρ' (or, equivalently, d) between any two elements of the finite set $F^{(3)}_{m+n}$. It follows from (7) that for every word $x \in F_{m+n}$, the value of $d(w, e)$ is at least $[l(w)/3]\delta$, where $l(w)$ denotes the reduced length of w.

9. Being free, the group F_{m+n} is residually finite, that is, admits a separating family of homomorphisms into finite groups. (Cf. e.g. [21], Ch. 7, exercise 5.) Therefore, for every natural k there exists a normal subgroup N_k such that the factor-group F_{m+n}/N_k is finite and the only word of reduced length $\leq k$ contained in N_k is identity. Denote by ϖ: $F_{m+n} \to F_{m+n}/N_k$ the factor-homomorphism and equip F_{m+n}/N_k with the factor-distance of d by letting

$$d'(g, h) = \inf\{d(w, v) \mid \varpi(w) = g, \varpi(v) = h\}.$$

10. It is immediate that d' is a bi-invariant pseudometric. Moreover, for $k \geq 3$ it is a metric: $d'(g, h) \geq \delta \geq \varepsilon/2$ whenever $g \neq h$, cf. step 8.

11. Now assume that $k \geq 3[\delta^{-1}] + 4$. Let $w, v \in F^{(3)}_{m+n}$ and $x, y \in N_k$ be arbitrary, $x \neq y$. Then $d(wx, vy) = d(e, x^{-1}w^{-1}vy) \geq 1$, because $w^{-1}v \in N_k$ and therefore $l(x^{-1}w^{-1}vy) \geq 3[\delta^{-1}]$ and (8) applies. Therefore, $d'(\varpi(w), \varpi(v)) = d(w, v)$ and the restriction of ϖ to $F^{(3)}_{m+n}$ is an isometry.

12. Now set $\tilde{X} = F_{m+n}/N_k$ and $\tilde{x}_j = \varpi(x_j)$. For every $i = 1, \ldots, n$, the inner automorphism of the finite metric group $\tilde{X}$ determined by $\varpi(g_i)$ is an isometry, because the metric is bi-invariant. Denote this isometry by $\tilde{g}_j$. The indexed metric spaces $(g_i x_j g_i^{-1})$ and $(\varpi(g_i)\tilde{x}_j\varpi(g_i)^{-1})$, $i = 1, \ldots, n, j = 1, \ldots, m$ are isometric by force of the concluding remark in (11). Taking into account (5), we finally conclude that the indexed metric spaces $(\tilde{g}_j \cdot \tilde{x}_i)$ and $(g_j \cdot x_i)$ are ε-isometric, as required. ∎

Remark 3.3: A careful analysis of the proof shows the existence of an absolute constant $C = C(m, n, \varepsilon) > 0$ such that the cardinality of the finite metric space in the statement of Theorem 3.2 does not exceed C.

3.2. Let G be a group. One can introduce a natural topology on the set of (equivalence classes) of all isometric actions of G on metric spaces (whose size has to be bounded from above; for instance, it is natural to consider actions on all metric spaces X of density character not exceeding the cardinality of G). This topology is similar to the Fell topology on the set of (equivalence classes) of unitary representations of a group (cf. [9] or [24], p. 12), and is even closer to

the topology introduced by Exel and Loring on the set of representations of a C^*-algebra [8].

A neighbourhood of an action τ of G by isometries on a metric space X is determined by the following set of data: a finite subset $F = \{g_1, \ldots, g_n\} \subseteq G$, an $\varepsilon > 0$, and a finite collection $X' = \{x_1, \ldots, x_m\} \subseteq X$. Say that an action ς of G on a metric space Y by isometries is in $V[F; X; \varepsilon](\tau)$, if for some finite collection $Y' = \{y_1, \ldots, y_n\} \subseteq Y$ the metric spaces $\{g_j \cdot x_i\}$ and $\{g_j \cdot y_i\}$, naturally indexed by $\{1, \ldots, m\} \times \{1, \ldots, n\}$, are ε-isometric. Our Theorem 3.2 can be now reformulated as follows.

COROLLARY 3.4: *Every action of a free group F on an arbitrary metric space by isometries is the limit of a net of actions of F by isometries on finite metric spaces.*

Remark 3.5: It is worth noting that approximation results of the above type are not unknown. For instance, as a corollary of a criterion by Exel and Loring [EL] and the known residual finite-dimensionality of the group C^*-algebras of free groups [GM], every representation of such a C^*-algebra in a Hilbert space is approximated in the Exel–Loring topology by finite-dimensional representations.

To cast the above result as one on approximation of *topological* groups, we need to remind the concept of the Urysohn universal metric space.

4. Urysohn metric spaces and their groups of isometries

We begin this Section with a summary of some known concepts and results from theory of Urysohn metric spaces (Subsections 4.1 and 4.2), after which we state a result on approximation of Polish topological groups by finite groups (Subsection 4.3), establish the fixed point on compacta property of the group of isometries of the complete separable Urysohn space (Subsection 4.4), and finally give a new proof of the fixed point on compacta propety for the infinite orthogonal groups (Subsection 4.5).

4.1. URYSOHN METRIC SPACES. A metric space X is called a (**generalized**) **Urysohn space** if it has the following property: whenever $A \subseteq X$ is a finite metric subspace of X and $A' = A \cup \{a\}$ is an arbitrtary one-point metric space extension of A, the embedding $A \hookrightarrow X$ extends to an isometric embedding $A' \hookrightarrow X$. (Cf. [43, 44, 27, 50, 47, 10] and [19], 3.11_+.)

There is only one, up to an isometry, complete separable Urysohn metric space, which we denote by $\mathbb{U}$. This space contains an isometric copy of every separable metric space. Moreover, if X is a separable metric space and $A \subseteq X$ is a finite subspace, then every isometric embedding $A \hookrightarrow \mathbb{U}$ extends to an isometric embedding $X \hookrightarrow \mathbb{U}$.

A metric space X is called **n-homogeneous**, where n is a natural number, if every isometry between two subspaces of X containing at most n elements each extends to an isometry of X onto itself. If X is n-homogeneous for every natural n, then it is said to be **ω-homogeneous**. The complete separable Urysohn space $\mathbb{U}$ is ω-homogeneous and moreover enjoys the stronger property: every isometry between two compact subspaces of X extends to an isometry of X onto itself. Other well-known metric spaces having the same higher homogeneity property are the unit sphere $\mathbb{S}$ of the infinite-dimensional Hilbert space $\mathcal{H}$ and the infinite-dimensional Hilbert space $\mathcal{H}$ itself ([3], Ch. IV, §38).

There are some obvious modifications of the concept of Urysohn metric space. For example, one can consider only metric spaces of diameter not exceeding a given positive number d. The corresponding complete separable Urysohn space will be denoted $\mathbb{U}_d$. Another possibility is to consider Urysohn metric spaces in the class of metric spaces whose metrics only take values in the lattice $\varepsilon\mathbb{Z}$, $\varepsilon > 0$. The corresponding object will be denoted $\mathbb{U}^{\varepsilon\mathbb{Z}}$ (respectively, $\mathbb{U}_d^{\varepsilon\mathbb{Z}}$).

Certainly, the above are not the only classes of metric spaces for which the Urysohn-type universal objects exist. For instance, the Urysohn metric spaces for the class of spherical metric spaces of a fixed diameter in the sense of Blumenthal [3] are spheres in spaces $l_2(\Gamma)$. The infinite-dimensional Hilbert spaces play the role of Urysohn metric spaces for the class of metric spaces embeddable into Hilbert spaces.

The following construction of the Urysohn space belongs to Katětov [27]. Let us say, following [27, 47, 10], that a 1-Lipschitz real-valued function f on a metric space X is **supported on,** or else **controlled by**, a metric subspace $Y \subseteq X$ if for every $x \in X$

$$f(x) = \inf\{\rho(x, y) + f(y)\colon y \in Y\}.$$

Put otherwise, f is the largest 1-Lipschitz function on X having the prescribed restriction to Y. For instance, every distance function $x \mapsto \rho(x, x_0)$ from a point x_0 is controlled by a singleton, $\{x_0\}$.

Let X be an arbitrary metric space. Denote by $\mathrm{E}(X)$ the collection of all functions $f\colon X \to \mathbb{R}$ controlled by some finite subset of X (depending on the

function) and having the property

$$|f(x) - f(y)| \leq d_X(x,y) \leq f(x) + f(y) \tag{4.1}$$

for all $x, y \in X$. If equipped with the supremum metric, $\mathrm{E}(X)$ becomes a metric space of the same density character as X, containing an isometric copy of X under the Kuratowski embedding:

$$X \ni x \mapsto [d_x\colon X \ni y \mapsto \rho(x,y) \in \mathbb{R}] \in \mathrm{E}(X).$$

Besides, the space $\mathrm{E}(X)$ contains all one-point metric extensions of every finite metric subspace of X.

One can form an increasing sequence of iterated extensions of the form

$$X, \mathrm{E}(X), \mathrm{E}^2 = \mathrm{E}(\mathrm{E}(X)), \ldots, \mathrm{E}^n(X) = \mathrm{E}(\mathrm{E}^{n-1}(X)), \ldots,$$

take the union, $\mathrm{E}^\infty(X)$, and form the metric completion of it, $\hat{\mathrm{E}}^\infty(X)$. The latter space is a generalized Urysohn space. If the metric space X is separable, then so is $\hat{\mathrm{E}}^\infty(X)$, and thus it is isometric to $\mathbb{U}$. If X is non-separable, then the resulting metric space $\hat{\mathrm{E}}^\infty(X)$ need not be ω-homogeneous.

If X is a separable metric space with $\mathrm{diam}(X) \leq d$ and throughout the above construction one replaces $\mathrm{E}(X)$ with the metric space $\mathrm{E}_d(X)$ formed by all functions f satisfying (4.1), bounded by d, and controlled by finite subspaces in a suitably modified sense, then the resulting metric space $\hat{\mathrm{E}}^\infty(X)$ is isometric to $\mathbb{U}_d$.

4.2. GROUPS OF ISOMETRIES. A remarkable feature of the above construction, discovered by Uspenskij, is that it enables one to keep track of groups of isometries.

Given an arbitrary metric space X, the topology of pointwise convergence and the compact-open topology on the group $\mathrm{Iso}(X)$ of all isometries of X onto itself coincide and turn $\mathrm{Iso}(X)$ into a Hausdorff topological group. The basic neighbourhoods of identity in this topology are of the form

$$V[F;\varepsilon] = \{g \in \mathrm{Iso}(X)\colon \forall x \in F,\ d_X(g(x),x) < \varepsilon\},$$

where $F \subseteq X$ is finite and $\varepsilon > 0$. If X is separable (and thus second-countable), then so is $\mathrm{Iso}(X)$.

Notice that in general the action of $\mathrm{Iso}(X)$ on the metric space X is not bounded (cf. Remark 2.17.2), while the action of $\mathrm{Iso}(X)$ by translations on the space of bounded uniformly continuous (or Lipschitz) functions on X, equipped with the supremum norm, is not, in general, continuous.

However, the isometric action of the group $\mathrm{Iso}(X)$ on the metric space of all 1-Lipschitz functions on X controlled by finite subsets happens to be continuous. Indeed, if a function $f \in E(X)$ is controlled by a finite $Y \subseteq X$, then the translation $g \circ f$ does not differ from f by more than ε at any point of X, provided $g \in V[Y;\varepsilon]$. Consequently, the canonical representation of $\mathrm{Iso}(X)$ in $\mathrm{E}(X)$ by isometries defines a topological group embedding $\mathrm{Iso}(X) \hookrightarrow \mathrm{Iso}(\mathrm{E}(X))$.

Iterating this process countably many times, one obtains a a continuous action of $\mathrm{Iso}(X)$ by isometries on $\mathrm{E}^\infty(X)$, which in its turn extends to a continuous action of $\mathrm{Iso}(X)$ on the metric completion $\hat{\mathrm{E}}^\infty(X) \cong \mathbb{U}$.

We adopt terminology suggested in [47] and say that a metric subspace Y is **g-embedded** into a metric space X if there exists an embedding of topological groups e: $\mathrm{Iso}(Y) \hookrightarrow \mathrm{Iso}(X)$ with the property that for every $h \in \mathrm{Iso}(Y)$ the isometry $e(h)$: $X \to X$ is an extension of h. The above argument establishes the following result.

PROPOSITION 4.1 (Uspenskij [45]): *Every separable metric space X can be g-embedded into the complete separable Urysohn metric space $\mathbb{U}$.*

Since every [second-countable] topological group G embeds into the isometry group of a suitable [separable] metric space [42], we arrive at the following.

THEOREM 4.2 (Uspenskij [45]): *The topological group $\mathrm{Iso}(\mathbb{U})$ is the universal second-countable topological group.*

(Cf. also [19], $3.11.\frac{2}{3}_+$.)

Since every isometry between two compact subspaces of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$ onto itself, we obtain the following useful corollary of Proposition 4.1.

COROLLARY 4.3: *Each isometric embedding of a compact metric space into $\mathbb{U}$ is a g-embedding.*

The question of the existence of universal topological groups of a given uncountable weight τ (in fact, of *any* uncountable weight τ) remains open. However, recently Uspenskij has established the following result.

THEOREM 4.4 ([47]): *Every topological group G embeds, as a topological subgroup, into the group of isometries $\mathrm{Iso}(X)$ of a suitable ω-homogeneous Urysohn metric space X of the same weight as G.*

The construction rather resembles the proof of Theorem 4.2, but in order to achieve ω-homogeneity of the union space, one alternates between the Katětov

metric extension $\mathrm{E}(\cdot)$ and the 'homogenization' extension, $H(\cdot)$, which forms the nontrivial technical core of the proof and is described in the following theorem.

THEOREM 4.5 (Uspenskij [47]): *Every metric space X g-embeds into an ω-homogeneous metric space $H(X)$ of the same weight as X.*

4.3. APPROXIMATION OF TOPOLOGICAL GROUPS. Now we can state yet another reformulation of the approximation Theorem 3.2.

THEOREM 4.6: *For every finite collection of isometries $g_1, \ldots, g_n$ of the complete separable Urysohn metric space $\mathbb{U}$ and every neighbourhood V of identity in* $\mathrm{Iso}(\mathbb{U})$ *there are isometries $h_1, \ldots, h_n \in \mathrm{Iso}(\mathbb{U})$ generating a finite subgroup and such that $h_i g_i^{-1} \in V$, $i = 1, \ldots, n$.*

Proof: One can assume that $V = V[X; \varepsilon]$, where $X = \{x_1, \ldots, x_m\} \subseteq \mathbb{U}$ and $\varepsilon > 0$. Using Theorem 3.2, choose a finite metric space $\tilde{X}$, elements $\tilde{x}_1, \ldots, \tilde{x}_m$ of $\tilde{X}$, and isometries $\tilde{g}_1, \ldots, \tilde{g}_n$ of $\tilde{X}$ such that the naturally indexed finite metric spaces

$$A = \{g_i^{-1} \cdot x_j \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

and

$$B = \{\tilde{g}_i^{-1} \cdot \tilde{x}_j \mid i = 1, 2, \ldots, n, j = 1, 2, \ldots, m\}$$

are $\varepsilon/2$-isometric.

Using Lemma 3.1, isometrically embed A and $\tilde{X}$ into a finite metric space Z in such a way that $d_Z(g_i^{-1}(x_i), \tilde{g}_i^{-1}(\tilde{x}_i)) \leq \varepsilon/2$ for all i, j. Now extend the embedding $A \hookrightarrow \mathbb{U}$ to an isometric embedding $Z \hookrightarrow \mathbb{U}$. According to Corollary 4.3, the (finite) group $\mathrm{Iso}(\tilde{X})$ simultaneously extends to a group of isometries of $\mathbb{U}$. Denote the extension of the isometry $\tilde{g}_i$ by h_i. One has for all i, j:

$$d(x_i, h_j g_j^{-1}(x_i)) = d(\tilde{g}_i^{-1}(x_i), g_i^{-1}(x_i)) \leq \varepsilon/2 < \varepsilon,$$

and the proof is finished. ∎

Let G be a group and let X be a metric space. Every action of G on X by isometries can be viewed as a homomorphism $\tau\colon G \to \mathrm{Iso}(X)$. Equip the set $\mathrm{Hom}(G, \mathrm{Iso}(X))$ of all such homomorphisms with the topology of pointwise convergence on G, that is, the one induced from the Tychonoff product $\mathrm{Iso}(X)^G$. Since, in its turn, the topological space $\mathrm{Iso}(X)$ is a subspace of the Tychonoff product X^X, one concludes that $\mathrm{Hom}(G, \mathrm{Iso}(X))$ is a topological subspace of the Tychonoff product $X^{G \times X}$. In this form, the identification of the collection of all actions $\tau\colon G \times X \to X$ with a subspace of $X^{G \times X}$ becomes obvious.

Call an action **periodic** if it factors through an action of a finite group. One can reformulate Theorem 4.6 as follows.

COROLLARY 4.7: *Let F be a free group. The set of periodic actions of F on the Urysohn metric space $\mathbb{U}$ is everywhere dense in the set of all actions.*

Let F_∞ denote the free group of countably infinite rank. The mapping associating to an action τ of F_∞ on $\mathbb{U}$ the closure of $\tau(F_\infty)$ in $\mathrm{Iso}(\mathbb{U})$ is a surjection from $\mathrm{Hom}(G, \mathrm{Iso}(X))$ onto the space $\mathcal{L}(\mathrm{Iso}(\mathbb{U}))$ of all closed subgroups of $\mathrm{Iso}(\mathbb{U})$. Equip the latter space with the corresponding quotient topology. The topology so defined satisfies the axiom T_0.

COROLLARY 4.8: *The set of finite subgroups is everywhere dense in the topological space $\mathcal{L}(\mathrm{Iso}(\mathbb{U}))$.*

This leads to an approximation result for Polish topological groups.

COROLLARY 4.9: *Let G be a Polish topological group. Then under every isomorphic embedding into* $\mathrm{Iso}(\mathbb{U})$ *the group G is the limit of a net of finite subgroups.*

Remark 4.10: At the first sight, the above may seem to contradict the general principle (in particular espoused and explained by Vershik in [49]) according to which approximability of an (infinite) group G by finite groups is essentially equivalent to amenability of G. In fact, our results are in perfect agreement with this principle in that the approximating groups come from 'without' the group G and thus form an approximation not to G itself, but to a suitable topological group extension of G, which indeed turns out to be amenable (and even extremely amenable).

4.4. THE FIXED POINT PROPERTY OF THE GROUP $\mathrm{Iso}(\mathbb{U})$. Theorems 4.6 and 2.2 enable us to deduce the fixed point on compacta property for the group of isometries of the complete separable Urysohn space $\mathbb{U}$.

THEOREM 4.11: *The group* $\mathrm{Iso}(\mathbb{U})$ *of all isometries of the complete separable Urysohn space $\mathbb{U}$, equipped with the standard (pointwise = compact-open) topology, is extremely amenable (has the fixed point on compacta property).*

Proof: Let the group $\mathrm{Iso}(\mathbb{U})$ act continuously on a compact space K. We will show that every finite collection of elements of $\mathrm{Iso}(\mathbb{U})$ has a common fixed point in K, from which the result follows by an obvious compactness argument. Fix an arbitrary such collection, $g_1, \ldots, g_n \in \mathrm{Iso}(\mathbb{U})$.

Let $U \in \mathcal{U}_K$ be an arbitrary element of the unique compatible uniform structure on K. Without loss in generality, assume that U is closed as a subset of $K \times K$

(and consequently compact). Using the boundedness of the action of $\mathrm{Iso}(X)$ on K, choose a finite $X \subseteq \mathbb{U}$ and an $\varepsilon > 0$ such that, whenever $g \in V = V[X;\varepsilon]$, one has $(g \cdot \kappa, \kappa) \in V$ for all $\kappa \in K$.

By Theorem 4.6, there are isometries $h_1, \ldots, h_n \in \mathrm{Iso}(\mathbb{U})$ generating a finite subgroup H and such that $h_i g_i^{-1} \in V$, $i = 1, \ldots, n$.

Let $\tilde{X}$ be a finite H-invariant subset of $\mathbb{U}$ containing X. The iterated Katětov extension $\hat{\mathrm{E}}^{\infty}(L_0(\mathbb{I}, \tilde{X}))$ contains $\tilde{X}$ as a subspace made up of all constant functions and is isometric to $\mathbb{U}$, and since $\tilde{X}$ is finite, an isometry between the two spaces can be chosen so as to extend the canonical embedding of $\tilde{X}$ into $\mathbb{U}$. Thus we obtain a chain of g-embeddings

$$\tilde{X} \subset L_0(\mathbb{I}, \tilde{X}) \subset \mathbb{U}.$$

The group $L_0(\mathbb{I}, H)$ acts on $L_0(\mathbb{I}, \tilde{X})$ continuously and isometrically (Lemma 2.18), and this action canonically extends to a continuous isometric action of the same group on the space $\hat{\mathrm{E}}^{\infty}(L_0(\mathbb{I}, \tilde{X})) \cong \mathbb{U}$. Thus we obtain a continuous group monomorphism $j\colon L_0(\mathbb{I}, H) \to \mathrm{Iso}(\mathbb{U})$ with the property that for every $h \in H$ one has $j(h)|_{\tilde{X}} = h|_{\tilde{X}}$.

Composing j with the action $\mathrm{Iso}(\mathbb{U}) \to \mathrm{Homeo}(K)$, we obtain a continuous action of $L_0(\mathbb{I}, H)$ on K. By force of Theorem 2.2, $L_0(\mathbb{I}, H)$ has a common fixed point in K, say κ. In particular, κ is fixed under the elements $j(h_1), \ldots, j(h_n) \in \mathrm{Iso}(\mathbb{U})$, where we identify elements of H with constants in $L_0(\mathbb{I}, H)$.

For all $x \in X$ and $i = 1, 2, \ldots, n$, one has

$$d(j(h_i)^{-1}(x), g_i^{-1}(x)) = d(h_i^{-1}(x), g_i^{-1}(x)) < \varepsilon$$

for all i and $x \in X$, implying that $j(h_i)g_i^{-1} \in V$ for $i = 1, 2, \ldots, n$. Consequently and by the choice of $V = V[X;\varepsilon]$,

$$(g_i\kappa, \kappa) \equiv (g_i\kappa, j(h_i)\kappa) \equiv (g_i\kappa, (j(h_i)g_i^{-1}) \cdot (g_i\kappa)) \in U$$

for all i. Denote by F_U the (non-empty) set of all points $x \in K$ with the property $(g_i x, x) \in U$ for all i. Since U is closed, so is $F_U \subseteq K$. If $U_1 \subseteq U_2$, then $F_{U_1} \subseteq F_{U_2}$. It means that $\{F_U\}$ is a centred system of closed subsets of the compact space K and therefore has a common point, which is clearly fixed under $g_1, \ldots, g_n$, as required. ∎

Remark 4.12: The same argument *verbatim* also establishes the fixed point on compacta property of the topological group $\mathrm{Iso}(\mathbb{U}_d)$ of isometries of the complete separable universal Urysohn space of finite diameter d.

4.5. A NEW PROOF OF THE FIXED POINT ON COMPACTA PROPERTY OF THE INFINITE ORTHOGONAL GROUP. The above proof can be easily modified so as to result in a new proof of extreme amenability of the orthogonal group $\mathrm{O}(\mathcal{H})$ of an infinite-dimensional Hilbert space with the strong operator topology. This proof does not rely on such advanced tools from geometry as Gromov's isoperimetric inequality for groups $\mathrm{SO}(n)$.

The following belongs to folklore.

LEMMA 4.13: *Let X be a metric subspace of the unit sphere $\mathbb{S}$ of a real Hilbert space $\mathcal{H}$. Suppose a topological group G acts on X continuously by isometries. Then the action of G extends to a strongly continuous action of G by isometries on the sphere $\mathbb{S}$ (that is, to a strongly continuous orthogonal representation of G in $\mathcal{H}$). Put otherwise, every metric subspace of the unit sphere $\mathbb{S}$ of a real Hilbert space is g-embedded into $\mathbb{S}$. If the linear span of X is dense in $\mathcal{H}$, the extension is unique.*

Proof: Since for every $x, y \in X$ the value of the inner product is uniquely determined by the Euclidean distance between the elements,

$$(x, y) = 1 - \frac{1}{2}\rho_X(x, y)^2,$$

there is only one way to turn the linear span $\mathrm{lin}(X)$ into a pre-Hilbert space so as to induce the given metric on X. The corresponding completion $\mathcal{K} = \widehat{\mathrm{lin}}(X)$ is isometrically isomorphic to the closed linear span of X in $\mathcal{H}$, that is, $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^\perp$. As another consequence of the same observation, every isometry of X lifts to a unique orthogonal transformation of $\mathcal{K}$. The resulting homomorphism $\pi\colon G \to \mathrm{O}(\mathcal{K})$ is continuous if the latter group is equipped with the topology of simple convergence on X or, which is the same, on $\mathrm{lin}(X)$. On the groups of isometries of metric spaces the topology of simple convergence on an everywhere dense subset coincides with the topology of simple convergence on the entire space. Consequently, the extended orthogonal representation π of G in $\mathcal{K}$ is strongly continuous. It remains to extend π to a representation $\begin{pmatrix} \pi & 0 \\ 0 & \mathrm{Id}_{\mathcal{K}^\perp} \end{pmatrix}$ of G in $\mathcal{H}$. The uniqueness statement is obvious. ∎

Here is an outline of the alternative proof of extreme amenability of $\mathrm{O}(\mathcal{H})_s$. We will be only considering the separable case $\mathcal{H} = l_2$; the extension to non-separable case is straightforward.

Every finite collection $g_1, g_2, \ldots, g_n$ of elements of $\mathrm{O}(l_2)$, viewed as isometries of the unit sphere $\mathbb{S}$, can be approximated (in the strong operator topology)

by a collection of elements $g'_1, g'_2, \ldots, g'_n$ of a finite-dimensional orthogonal subgroup in the following sense: for a given natural m and an $\varepsilon > 0$, one has $\|g_i(e_j) - g'_i(e_j)\| < \varepsilon$ for all $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, where $g'_j \in \mathrm{O}(N)$, e_j denote the standard basic vectors, and the rank N is sufficiently large.

According to Lemma 2.18, the topological group $L_0(\mathbb{I}, \mathrm{O}(N))$ acts continuously by isometries on the metric space $L_2(\mathbb{I}, \mathbb{S}^N)$, equipped with the l_2-metric. (The topology induced on $L(\mathbb{I}, \mathbb{S}^N)$ by l_2-metric is still that of convergence in measure, because $\mathbb{S}^N$ is compact.) The metric space $L_2(\mathbb{I}, \mathbb{S}^N)$ is spherical of diameter one and thus can be embedded into $\mathbb{S}$ as a metric superspace of $\mathbb{S}^N$. Using Lemma 4.13, we obtain a chain of continuous monomorphisms of topological groups

$$\mathrm{O}(N) < L_0(\mathbb{I}, \mathrm{O}(N)) < \mathrm{Iso}(L_2(\mathbb{I}, \mathbb{S}^N)) < \mathrm{O}(l_2).$$

According to Theorem 2.2, the second topological group on the left is extremely amenable. It follows that the orthogonal operators $g'_1, g'_2, \ldots, g'_n$ have a common fixed point in every compact space upon which $\mathrm{O}(l_2)$ acts continuously. Now the proof is accomplished in the same way as in Theorem 4.11.

5. Ramsey-type theorems for metric spaces vs f.p.c. property

5.1. RAMSEY–DVORETZKY–MILMAN PROPERTY. In order to extend the result about fixed point on compacta property of the isometry group $\mathrm{Iso}(\mathbb{U})$ beyond the separable case, we will obtain a new characterization of extremely amenable groups of isometries in terms of a Ramsey-type property of the metric spaces X.

The following is an adaptation from [18], Sect. 9.3.

Definition 5.1: Let G be a group of uniform isomorphisms of a uniform space X. We will say that the pair (G, X) has the **Ramsey–Dvoretzky–Milman property** if for every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$, and every compact $K \subseteq X$, the function f is ε-constant on a suitable translate of K, that is, there is a $g \in G$ such that

$$\mathrm{Osc}(f \mid gK) < \varepsilon.$$

Equivalently, 'compact' can be replaced with 'finite.'

We defer two master examples (Ex. 5.6 and 5.8) in order to precede them by a few simple preliminary results. The following is established by pulling back the function f from Y to X.

LEMMA 5.2: *Let G be a group, acting by uniform isomorphisms on the uniform spaces X and Y, and let $f\colon X \to Y$ be an equivariant uniformly continuous map with everywhere dense range. If the pair (G, X) has the Ramsey–Dvoretzky–Milman property, then so does (G, Y).*

Denote by $\mathcal{U}_X^*$ the totally bounded replica of the uniform structure $\mathcal{U}_X$ on X, that is, the coarsest uniform structure preserving the uniform continuity of every bounded uniformly continuous function on X. Basic entourages of the diagonal for $\mathcal{U}_X^*$ are of the form

$$\{(g,h) \in X \times X\colon |f(x) - f(y)| < \varepsilon\},$$

where $f\colon X \to \mathbb{R}^N$ is bounded uniformly continuous, $N \in \mathbb{N}$.

The following reformulation of the R–D–M property is immediate.

PROPOSITION 5.3: *A pair (G, X) has the Ramsey–Dvoretzky–Milman property if and only if for every compact (equivalently: finite) $K \subseteq X$ and every entourage $V \in \mathcal{U}_X^*$ there is a $g \in G$ with gK being V-small: $gK \times gK \subseteq V$.*

PROPOSITION 5.4: *Let $X = (X, \mathcal{U}_X)$ be a uniform space. A basis of entourages for the totally bounded replica $\mathcal{U}_X^*$ of $\mathcal{U}_X$ is given by all finite covers of the form $\{V[A]\colon A \in \gamma\}$, where γ is an arbitrary finite cover of X and $V \in \mathcal{U}_X$.*

Proof: The claim consists of two parts: first, that all sets of the form

$$\bigcup_{A \in \gamma} V[A] \times V[A], \ \gamma \text{ finite}, \ V \in \mathcal{U}_X$$

are elements of $\mathcal{U}_X^*$, and second, that each enourage from $\mathcal{U}_X^*$ contains a set of the above type.

(1) Given γ, V, and A as above, choose a bounded uniformly continuous pseudometric d on X such that $(d(x,y) < 1) \Rightarrow ((x,y) \in V)$, and introduce a bounded uniformly continuous function f from X to the Euclidean space $\mathbb{R}^{|\gamma|}$ with each component f_A, $A \in \gamma$, defined by

$$X \ni x \mapsto f_A(x) := d(x, A) \in \mathbb{R}.$$

The set $\{(x,y) \in X^2\colon |f(x) - f(y)| < 1\}$ is an element of $\mathcal{U}_X^*$ and a subset of $\bigcup_{A \in \gamma} V[A] \times V[A]$.

(2) Let $W \in \mathcal{U}_X^*$ be arbitrary. Choose a bounded uniformly continuous function $f\colon X \to \mathbb{R}^N$ and an $\varepsilon > 0$ such that $\{(x,y) \in X^2\colon |f(x) - f(y)| < \varepsilon\} \subseteq W$.

Partition the image $f(X)$ into finitely many pieces of diameter $\leq \varepsilon/2$ each and let γ be the family of preimages of those pieces under f. Define

$$V = \{(x,y) \in X^2 \colon |f(x) - f(y)| < \varepsilon/2\} \in \mathcal{U}_X^* \subseteq \mathcal{U}_X.$$

Clearly, $\bigcup_{A\in\gamma} V[A] \times V[A] \subseteq W$. ∎

As an immediate corollary, one obtains the following.

PROPOSITION 5.5: *A pair (G,X) has the Ramsey–Dvoretzky–Milman property if and only if for every compact (equivalently: finite) $K \subseteq X$, every finite cover γ of X, and every entourage $V \in \mathcal{U}_X$, there is a $g \in G$ such that gK is contained in the V-neighbourhood of some $A \in \gamma$.*

Here is the first major example.

Example 5.6: Let Γ be an infinite set, and let n be a natural number. Choose as G the group S_Γ^f of all finite permutations of Γ, and as X the set $\Gamma^{(n)}$ of all n-subsets of Γ, equipped with the finest (discrete) uniformity. Using Proposition 5.5, one can easily see that the pair $(\Gamma^{(n)}, S_\Gamma^f)$ has the Ramsey–Dvoretzky–Milman property, which statement is indeed equivalent to the finite Ramsey theorem.

Recall that the basic entourages for the left uniform structure $\mathcal{U}_\natural(G)$ on a topological group G are of the form

$$V_\natural = \{(g,h) \in G \times G \colon g^{-1}h \in V\},$$

where V is a neighbourhood of identity in G. If d is a left invariant continuous pseudometric on G and $\varepsilon > 0$, then the set $V[d;\varepsilon] = \{(x,y) \in X^2 \colon d(x,y) < \varepsilon\}$ is an element of $\mathcal{U}_\natural(G)$. Since for every neighbourhood of identity V there is a bounded left invariant continuous pseudometric d on G with $(d(x,e_G) < 1) \Rightarrow (x \in V)$ and consequently $V_\natural \supseteq V[d;1]$, it follows that the left uniform structure on a topological group is determined by left invariant bounded continuous pseudometrics.

If d is a left invariant continuous pseudometric on G, then

$$H_d = \{x \in G \colon d(x,e_G) = 0\}$$

forms a closed subgroup of G, and the pseudometric d induces a continuous left-invariant metric $\hat{d}$ on the factor-space G/H_d by the formula $\hat{d}(xH,yH) := d(x,y)$. The canonical factor-map $\pi \colon G \to (G/H_d, \hat{d})$ is uniformly continuous. Notice that in general both the topology and the uniform structure induced by $\hat{d}$ are coarser

than the factor-topology and the left uniform structure on G/H_d. We will denote the G-space G/H_d equipped with the left invariant metric $\hat{d}$ by G/d, which is consistent with the notation sometimes used in set-theoretic topology: in our situation, G/d is the metric space canonically associated to the pseudometric space (G, d).

The following result (which grew out of V.V. Uspenskij's conjecture) reveals the link between the Ramsey–Dvoretzky–Milman property and the existence of fixed points.

THEOREM 5.7: *For a topological group G, the following are equivalent.*

(i) *G has the fixed point on compacta property.*

(ii) *The pair $(G, G_\natural)$ has the Ramsey–Dvoretzky–Milman property.*

(iii) *For every left-invariant continuous pseudometric d on G, the pair $(G, G/d)$ has the Ramsey–Dvoretzky–Milman property.*

(iv) *Whenever G acts continuously and transitively by isometries on a metric space X, the pair (G, X) has the Ramsey–Dvoretzky–Milman property.*

(v) *For some family D of bounded continuous left invariant pseudometrics d, generating the topology of G, each pair $(G, G/d)$ has the Ramsey–Dvoretzky–Milman property.*

Proof: (i) $\Leftrightarrow$ (ii): according to Theorem 2.8, the fixed point on compacta property of a topological group G is equivalent to the following: for every bounded *right* uniformly continuous function f on G taking values in a finite-dimensional Euclidean space, every finite collection of elements $g_1, g_2, \ldots, g_n \in G$, and every $\varepsilon > 0$, there is an $x \in G$ such that

$$|f(x) - f(g_i x)| < \varepsilon \quad \text{for all } i = 1, 2, \ldots, n.$$

The mirror image of the above statement applies to *left* uniformly continuous functions and calls for the existence of an $x \in G$ with the property

$$|f(x) - f(x g_i)| < \varepsilon \quad \text{for all } i.$$

This amounts to the Ramsey–Dvoretzky–Milman property for the pair $(G, G_\natural)$ relative to the left action (with $K = \{e_G, g_1, g_2, \ldots, g_n\}$).

(ii) $\Rightarrow$ (iii): as the canonical map $G \to G/d$ is uniformly continuous and G-equivariant, Lemma 5.2 applies.

(iii) $\Rightarrow$ (iv): Let d_X denote the invariant metric on X. Fix an arbitrary point $x_0 \in X$. The formula $d(g, h) := d_X(gx_0, hx_0)$ defines a left invariant continuous

pseudometric on G, and the map $G \ni g \mapsto gx_0 \in X$ factors through to a G-equivariant isometric isomorphism between G/d and X.

(iv) $\Rightarrow$ (v): Trivial, as G acts on each space G/d continuously and transitively by isometries.

(v) $\Rightarrow$ (ii): Suppose we are given a finite subset $F \subseteq G$, a finite cover γ of G, and a basic element V_η of the left uniformity G_η, where V is a neighbourhood of identity in G. Choose a bounded left invariant continuous pseudometric $d \in D$ with the property $(d(x, e_G) < 1) \Rightarrow (x \in V)$. The sets $\pi(A)$, $A \in \gamma$, where $\pi\colon G \to G/d$ is the factor-map, form a finite cover of G/d, and by assumption there is a $g \in G$ such that $g\pi(F)$ is entirely contained in the 1-neighbourhood of some $\pi(A)$, $A \in \gamma$. Denote, as before, $H_d = \{g \in G\colon d(g, e_G) = 0\}$. The value of d is independent on the choice of representatives in left cosets: $d(xh_1, yh_2) = d(x, y)$ for all $x, y \in G$, $h_1, h_2 \in H_d$. Let $f \in F$ be any. Since $\hat{d}(g\pi(f), \pi(a)) < 1$ for some $a \in A$, one has $d(gf, a) < 1$, that is, gF is contained in $V_\eta[A] = AV$, and the Ramsey–Dvoretzky–Milman property of (G, G_η) is thus verified. ∎

Example 5.8: The second major example is given by the pair consisting of the full unitary group $U(\mathcal{H})$ of an infinite-dimensional Hilbert space $\mathcal{H}$ and the unit sphere $\mathbb{S}_\mathcal{H}$ equipped with the Euclidean distance. The Ramsey–Dvoretzky–Milman property of this pair follows from Theorem 5.7 and the extreme amenability of $U(\mathcal{H})_s$ (cf. Subsection 4.5). In fact, a direct proof of this property does not require the extreme amenability of the unitary group, and such was the original proof by Milman [31] (who then used the R–D–M property to give a new proof of Dvoretzky theorem on almost spherical sections of convex bodies), cf. also [18], Sect. 9.3.

For sufficiently homogeneous spaces and their full groups of isometries Theorem 5.7 assumes a combinatorial form of a Ramsey-type result for metric spaces somewhat in the spirit of [30] or [29], but in an 'approximate' implementation. We proceed to examine this connection now.

5.2. RAMSEY-TYPE PROPERTIES OF METRIC SPACES. Let X be a metric space, and let F be a finite metric subspace of X. The stabilizer of F,

$$\mathrm{St}_F = \{g \in \mathrm{Iso}(X)\colon gx = x \text{ for each } x \in F\},$$

is a closed subgroup of $\mathrm{Iso}(X)$. Denote by $X^{\hookleftarrow F}$ the family of all isometric embeddings of F into X, equipped with the natural action of $\mathrm{Iso}(X)$ on the left:

$$X^{\hookleftarrow F} \ni j \mapsto g \circ j \in X^{\hookleftarrow F}.$$

The supremum metric on $X^{\hookleftarrow F}$, given by

$$d_{sup}(i,j) = \max\{d(i(x),j(x)) \colon x \in F\},$$

is $\mathrm{Iso}(X)$-invariant. Denote by d_F the pull-back of the metric d_{sup} to $\mathrm{Iso}(X)$:

$$d_F(g,h) := d_{sup}(gi_F, hi_F),$$

where $i_F \colon F \hookrightarrow X$ is the canonical embedding. Left-invariant pseudometrics of the form d_F, where F runs over all finite subspaces of X, generate the usual topology of pointwise convergence on $\mathrm{Iso}(X)$.

If X is $|F|$-homogeneous, the following establishes an isomorphism of $\mathrm{Iso}(X)$-sets:

$$G/\mathrm{St}_F \ni g\,\mathrm{St}_F \mapsto [g|_F \colon F \to gF] \in X^{\hookleftarrow F}. \tag{5.1}$$

In the combinatorial spirit, we will refer to [finite] partitions of a metric space X as **colourings** of X [using finitely many colours]. A subset $Y \subseteq X$ is **monochromatic** if $Y \subseteq X$ for some $A \in \gamma$, and **monochromatic up to an** $\varepsilon > 0$ if Y is contained in the ε-neighbourhood of some $A \in \gamma$.

A direct application of Theorem 5.7 now results in the following.

THEOREM 5.9: *Let X be an ω-homogeneous metric space. The following conditions are equivalent.*

(i) *The full group of isometries* $\mathrm{Iso}(X)$ *with the pointwise topology is extremely amenable.*

(ii) *Let $F \subseteq X$ be a finite metric space, and let $X^{\hookleftarrow F}$ be coloured using finitely many colours. Then for every finite metric subspace $G \subseteq X$ and every $\varepsilon > 0$ there is an isometric copy of G, $G' \subseteq X$, such that all isometric embeddings $F \hookrightarrow X$ that factor through G' are monochromatic up to ε.*

Remark 5.10: There is a natural surjection from $X^{\hookleftarrow F}$ onto the collection $X^{(F)}$ of all subspaces of X isometric to F, as the latter space is obtained from the former one by factoring out the group of distance-preserving permutations of F:

$$X^{(F)} \cong X^{\hookleftarrow F}/\mathrm{Iso}(F).$$

In particular, if the metric space F is rigid (for example, if no two distances between different pairs of points are the same), then the spaces $X^{(F)}$ and $X^{\hookleftarrow F}$ can be identified. In general, however, the distinction between the two spaces has to be maintained, and as we shall see (Theorem 6.9), some groups of isometries of ω-homogeneous metric spaces fail to have the fixed point on compacta property namely due to the fact that the two spaces $X^{(F)}$ and $X^{\hookleftarrow F}$ are different.

Remark 5.11: Theorem 5.9 provides at one's disposal a rather versatile tool. The main application in this article will be to establish the extreme amenability of groups of isometries of ω-homogeneous generalized Urysohn spaces. The result shall also be used to demonstrate that some groups of isometries are not extremely amenable. And finally, one can turn Theorem 5.9 around in order to deduce Ramsey-type results for metric spaces from the known results on extreme amenability of various topological groups established by other means. The next Section contains examples of applications of each sort.

6. Applications

6.1. EXTREME AMENABILITY OF THE GROUPS $\mathrm{Iso}(U)$. We want to formalize the content of the condition (ii) of Theorem 5.9, as follows.

Definition 6.1: Let F and G be finite metric spaces, let $m \in \mathbb{N}_+$, and let $\varepsilon > 0$. Denote by $R(F, G, m, \varepsilon)$ the following property of a metric space X:

$X \in R(F, G, m, \varepsilon) \Leftrightarrow$ *for every colouring of the set* $X^{\hookleftarrow F}$ *of all isometric embeddings of* F *into* X *with* $\leq m$ *colours, there is an isometric embedding* $j\colon G \hookrightarrow X$ *such that all embeddings of* F *into* X *that factor through* j *are monochromatic up to* ε.

Say that a metric space X has property R if $X \in R(F, G, m, \varepsilon)$ for all finite metric spaces F, G embeddable into X, for all $m \in \mathbb{N}$, and all $\varepsilon > 0$.

Remark 6.2: Now Theorem 5.9 can be reformulated as follows: *an* ω*-transitive metric space* X *has property* R *if and only if the topological group* $\mathrm{Iso}(X)$ *is extremely amenable.*

PROPOSITION 6.3: *Let* F *and* G *be finite metric spaces, let* X *be a metric space containing a copy of* F*, let* m *be a natural number, and let* $\varepsilon > 0$*. The following are equivalent.*

(i) $X \in R(F, G, m, \varepsilon)$.

(ii) *There is a finite subspace* $Z \subseteq X$ *containing a copy of* F *such that* $Z \in R(F, G, m, \varepsilon)$.

Proof: (i) $\Rightarrow$ (ii): assume $\neg$(ii), that is, no finite subspace Z of X containing a copy of F is in $R(F, G, m, \varepsilon)$. Denote by $\mathcal{Z}$ the collection of all finite metric subspaces $Z \subseteq X$ with $Z^{\hookleftarrow F} \neq \emptyset$. By assumption, $\mathcal{Z} \neq \emptyset$.

Then for every $Z \in \mathcal{Z}$ the set $Z^{\hookleftarrow F}$ admits a colouring with m colours, which we will view as a function $f_Z\colon Z^{\hookleftarrow F} \to \{1, 2, \ldots, m\}$, in such a way that the following holds:

(*) for every isometric embedding $i\colon G \hookrightarrow Z$ and every colour $k = 1, 2, \ldots, m$ there is an isometric embedding $j_k\colon F \hookrightarrow G$ such that the ε-neighbourhood of $i \circ j_k$ in $Z^{\hookleftarrow F}$ contains no elements of colour k.

The system $\mathcal{Z}$ is directed by inclusion, and the collection of intervals $[K, \infty) = \{Z \in \mathcal{Z}\colon K \subseteq Z\}$, where $K \subseteq X$ is finite, is a filter on $\mathcal{Z}$, which we will denote by $\mathcal{F}$. Since X can be assumed infinite (otherwise there is nothing to prove), $\mathcal{F}$ extends to a free ultrafilter Λ on $\mathcal{Z}$. For every $j \in X^{\hookleftarrow F}$, one has $[\{j(F)\}, \infty) \in \mathcal{F} \subset \Lambda$, and therefore exactly one of the sets $\{Z \in \mathcal{Z}\colon f_Z(j) = i\}$, $1 \leq i \leq m$ is in Λ. Consequently, the function

$$f(j) = \lim_{\Lambda} f_Z(j)$$

determines a colouring of $X^{\hookleftarrow F}$ with m colours.

Now let $\iota\colon G \hookleftarrow X$ be an arbitrary isometric embedding, and let $k \in \{1, 2, \ldots, m\}$ be a colour. For every $Z \in [\iota(G), \infty)$ choose, using (*), an isometric embedding $j_{Z,k}\colon F \hookrightarrow G$ with no element in the ε-neighbourhood of $\iota \circ j_{Z,k}$, formed in $Z^{\hookleftarrow F}$, being of f_Z-colour k. For every $x \in F$ define $j_k(x) = \lim_\Lambda j_{Z,k}(x) \in G$. (The metric space G is finite.) This j_k is an isometric embedding of F into G with the property that the ε-neighbourhood of $\iota \circ j_k$ formed in all of $X^{\hookleftarrow F}$ contains no elements of colour k. Thus, $\neg$(i) is established.

(ii) $\Rightarrow$ (i): evident. ∎

COROLLARY 6.4: *Let X and Y be two metric spaces having, up to isometry, the same finite metric subspaces. If X has property R, then so does Y.*

THEOREM 6.5: *Let X and Y be two ω-homogeneous metric spaces, having, up to isometry, the same finite metric subspaces. Then the topological group* $\mathrm{Iso}(X)$ *has the fixed point on compacta property if and only if the topological group* $\mathrm{Iso}(Y)$ *does.*

Proof: Combine Theorem 5.9 and Corollary 6.4. ∎

We can finally deduce from Theorem 6.5 and Theorem 4.11 the following result, which is the *raison d'être* of the article.

THEOREM 6.6: *Let U be a generalized Urysohn metric space. If U is ω-homogeneous, then the group* $\mathrm{Iso}(U)$ *has the fixed point on compacta property.*

Modulo Uspenskij's Theorem 4.4, the above Theorem implies the following.

COROLLARY 6.7: *Every topological group embeds, as a topological subgroup, into an extremely amenable topological group, that is, a topological group with the fixed point on compacta property.*

Even the following appears to be a new result.

COROLLARY 6.8: *Every topological group embeds, as a topological subgroup, into an amenable topological group.*

6.2. GROUPS OF ISOMETRIES OF DISCRETE URYSOHN SPACES. Here we will demonstrate how Theorem 5.9 can be used to show the absence of the fixed point on compacta property in the case where the ω-homogeneous metric space in question fails the 'strong' version of Ramsey-type property.

THEOREM 6.9: *The group of isometries of the discrete Urysohn metric space $\mathbb{U}^{\varepsilon\mathbb{Z}}$ does not have the fixed point on compacta property.*

Proof: Denote by $\{a, b\}$ the two-element metric space with $d(a, b) = \varepsilon$. Partition the set $(\mathbb{U}^{\varepsilon\mathbb{Z}})^{\leftarrow\{a,b\}}$ of all isometric embeddings of $\{a, b\}$ into $\mathbb{U}^{\varepsilon\mathbb{Z}}$ into two disjoint subsets A, B in such a way that whenever an injection i: $\{a, b\} \hookrightarrow (\mathbb{U}^{\varepsilon\mathbb{Z}})$ is in A, the 'flip' injection $i \circ \sigma_2$ is in B, and vice versa. Since the space $\mathbb{U}^{\varepsilon\mathbb{Z}}$ is ε-discrete, the ε-neighbourhood of a subset X is X itself, and 'monochromatic up to ε' means in this context simply 'monochromatic.' One concludes that, with respect to the colouring $\{A, B\}$, no pair of injections of the form $Y = \{i, i \circ \sigma_2\}$ is monochromatic up to ε, and thus the metric space $(\mathbb{U}^{\varepsilon\mathbb{Z}})^{\leftarrow\{a,b\}}$, upon which the group $\mathrm{Iso}(\mathbb{U}^{\varepsilon\mathbb{Z}})$ acts transitively and continuously by isometries, fails the Ramsey–Dviretzky–Milman property. ∎

Remark 6.10: The same result holds for discrete Urysohn spaces of bounded diameter, $\mathbb{U}_d^{\varepsilon\mathbb{Z}}$. In particular, letting $\varepsilon = 1 = d$, we obtain a result proved by the present author in [37], Th. 6.5: the group of permutations S_∞ of an infinite set, equipped with the pointwise topology, is not extremely amenable. (This result seems to answer in the negative an old question by Furstenberg discussed in [20].)

Notice also that the groups of isometries of infinite, ω-homogeneous metric spaces need not be extremely amenable.

The countable metric space $\mathbb{U}_1^{\mathbb{Z}}$, equipped with the $\{0, 1\}$-valued metric, actually satisfies a 'weaker' version of the Ramsey result, namely the one for finite *subspaces*, rather than for their *injections*, and this result is the well-known Finite Ramsey Theorem. (Cf. Ex. 5.6.) However, as we have just seen, the group

fails the 'stronger' version for embeddings of finite spaces! The latter circumstance destroys the extreme amenability of S_∞.

Finally notice that the topological group S_∞ is amenable, because it is approximated from within by an increasing chain of finite groups of permutations whose union is everywhere dense.

6.3. DEDUCING RAMSEY-TYPE THEOREMS FOR METRIC SPACES. By force of Theorem 5.9, the immediate corollary — and in fact an equivalent form — of the fixed point on compacta property of the group $\mathrm{Iso}(\mathbb{U})$ (Theorem 4.11) is the following Ramsey-type result.

COROLLARY 6.11: *Let F be a finite metric space, and let all isometric embeddings of F into $\mathbb{U}$ be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbb{U}$ of G such that all isometric embeddings of F into $\mathbb{U}$ that factor through G are monochromatic up to ε.*

By restricting ourselves to considering only $\mathrm{Iso}(F)$-invariant collections of embeddings of F into $\mathbb{U}$, we arrive at the following.

COROLLARY 6.12: *Let F be a finite metric space. Let all subspaces of the Urysohn space $\mathbb{U}$ isometric to F be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is a subspace $G' \subseteq \mathbb{U}$ isometric to G whose subspaces isometric to F are monochromatic up to ε.*

Applications to spherical spaces are probably more interesting. (Cf. comments in [30] at the bottom of p. 460). The unit sphere of the infinite-dimensional Hilbert space $\mathcal{H}$ is an ω-homogeneous metric space, and the orthogonal group of $\mathcal{H}$ with the strong operator topology (that is, the topology of simple convergence on the sphere) is extremely amenable [20]. As a corollary, we obtain Ramsey-type results for the Hilbert sphere.

COROLLARY 6.13: *Let F be a finite metric subspace of the unit sphere $\mathbb{S}^\infty$ in an infinite-dimensional Hilbert space. Let all isometric embeddings of F into $\mathbb{S}^\infty$ be coloured using finitely many colours. Then for every finite metric subspace G of the sphere and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbb{S}^\infty$ of G such that all isometric embeddings of F into G' are monochromatic up to ε.*

COROLLARY 6.14: *Let F be a finite metric subspace of the unit sphere $\mathbb{S}^\infty$ in an infinite-dimensional Hilbert space. Let all subspaces of $\mathbb{S}^\infty$ isometric to F be coloured using finitely many colours. Then for every finite subspace Y of*

the sphere and every $\varepsilon > 0$ there is a subspace $Y' \subseteq \mathbb{S}^\infty$ isometric to Y whose subspaces isometric to F are monochromatic up to ε.

To establish similar corollaries for metric subspaces of the infinite-dimensional Hilbert space, we need the following result. Notice that amenability of the group $\mathrm{Iso}(\mathcal{H})$ of affine isometries of a Hilbert space [24], p. 47.

THEOREM 6.15: *The group $\mathrm{Iso}(\mathcal{H})$ of affine isometries of a Hilbert space $\mathcal{H}$ of infinite dimension is extremely amenable.*

Proof: The topological group $\mathrm{Iso}(\mathcal{H})$ is isomorphic to the semidirect product $\mathrm{O}(\mathcal{H}) \ltimes \mathcal{H}$ of the full orthogonal group $\mathrm{O}(\mathcal{H})$ equipped with the strong operator topology and the additive group of the Hilbert space $\mathcal{H}$ with the usual norm topology, formed with respect to the natural action of $\mathrm{O}(\mathcal{H})$ on $\mathcal{H}$ by rotations. (Cf. [24].) Suppose $\mathrm{Iso}(\mathcal{H})$ acts continuously on a compact space K. Since the group $\mathrm{O}(\mathcal{H})$ (identified with a subgroup of $\mathrm{Iso}(\mathcal{H})$) is extremely amenable ([20]; cf. also Subsection 4.5), it has a fixed point $\kappa \in K$. The mapping $\mathcal{H} \ni x \mapsto x \cdot \kappa \in K$, where $\mathcal{H}$ is viewed as a closed normal subgroup of $\mathrm{Iso}(\mathcal{H})$, is $\mathrm{Iso}(\mathcal{H})$-equivariant, continuous, and has everywhere dense image in K, and thus K is an equivariant $\mathrm{Iso}(\mathcal{H})$-compactification of the homogeneous factor-space $\mathcal{H} \cong \mathrm{Iso}(\mathcal{H})/\mathrm{O}(\mathcal{H})$.

Let $\varphi\colon K \to \mathbb{R}^N$ be an arbitrary continuous function, $N \in \mathbb{N}$. Its pull-back, $f(x) =: \varphi(x \cdot \kappa)$, to $\mathcal{H}$ is right uniformly continuous. (A standard result in abstract topological dynamics.) If $\varepsilon > 0$ is arbitrary, then for some neighbourhood $V = V[F; \delta]$ of identity in $\mathrm{Iso}(\mathcal{H})$ one has $|f(g(0)) - f(h(0))| < \varepsilon$ whenever $gh^{-1} \in V$. Without loss in generality and slightly perturbing the points of F if necessary, one can assume that elements of F are affinely independent. Let $x, y \in \mathcal{H}$ be two arbitrary elements with the property $\|x - z\| = \|y - z\|$ for each $z \in F$. Find an isometric copy of F, say F', such that $F' \cup \{0\}$ is isometric to $F \cup \{x\}$ (or, equivalently, to $F \cup \{y\}$). There is an isometry g of $\mathcal{H}$ taking $F' \cup \{0\}$ to $F \cup \{x\}$, and an isometry h taking $F' \cup \{0\}$ to $F \cup \{y\}$. In particular, $gh^{-1}|_F = \mathrm{Id}_F \in V$, and consequently $|f(x) - f(y)| < \varepsilon$. Thus, the function f is ε-constant on every affine sphere of codimension $|F|$ having the form

$$\{x \in \mathcal{H}\colon \|x - z\| = r_z, z \in F\} \equiv \bigcap_{z \in F} \mathbb{S}_{r_z}(z).$$

Another way to say it is that, up to ε, the function $f(x)$ only depends on the collection of distances $\{\|x - z\| : z \in F\}$.

Now let $g_1, \ldots, g_n \in \mathrm{Iso}(\mathcal{H})$ be an arbitrary collection of isometries. By slightly perturbing them if necessary, one can assume without loss in generality that all

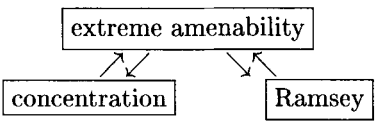
the vectors z and $g_i^{-1}(z)$, $z \in F$, $i = 1, 2, \ldots, n$, are affinely independent. Because of infinite-dimensionality of $\mathcal{H}$, every element x of some affine subspace of $\mathcal{H}$ of finite codimension has the property that for every $i = 1, 2, \ldots, n$ and each $z \in F$, one has $\|x - g_i^{-1}(z)\| = \|x - z\|$. Fix any such x. Then the values of f at the points $x, g_1(x), g_2(x), \ldots, g_n(x)$ differ by less than ε. Now we can apply Theorem 2.8 to conclude that K has a fixed point for $\mathrm{Iso}(\mathcal{H})$. ∎

COROLLARY 6.16: *Let F be a finite metric subspace of the infinite-dimensional Hilbert space $\mathcal{H}$. Let all isometric embeddings of F into $\mathcal{H}$ be coloured using finitely many colours. Then for every finite collection Y of such embeddings and every $\varepsilon > 0$ there is a collection of embeddings Y' congruent to Y and monochromatic up to ε.*

COROLLARY 6.17: *Let F be a finite metric subspace of an infinite-dimensional Hilbert space $\mathcal{H}$. If all subspaces of $\mathcal{H}$ isometric to F are coloured using finitely many colours, then for every finite subspace G of $\mathcal{H}$ and every $\varepsilon > 0$ there is an isometric copy G' of G in $\mathcal{H}$ such that all subspaces of G' isometric to F are monochromatic up to ε.*

7. Concluding remarks

In this article we have investigated some relationships inside the following triangle:



Deeper explorations of the Ramsey–Milman phenomenon in topological transformation groups require discovering situations in which a 'phase transition' between concentration and dissipation occurs in families of topological groups/dynamical systems. (Cf. [4].) It could be, for example, that a solution to Glasner's problem on the existence of a minimally almost periodic group topology on the integers without the fixed point on compacta property [12] lies namely in this direction.

In connection with the Banach–Mazur problem (cf. [6]), it could be worth investigating the fixed point on compacta property for the groups of isometries of separable Banach spaces admitting a transitive norm.

Finally, we do not know if the results of Section 6 can be put in direct connection with the Euclidean Ramsey theory [16].

References

[1] J. Auslander, *Minimal Flows and Their Extensions*, North-Holland Mathematics Studies **153**, North-Holland, Amsterdam–New York–London–Tokyo, 1988.

[2] W. Banaszczyk, *On the existence of exotic Banach–Lie groups*, Mathematische Annalen **264** (1983), 485–493.

[3] L. M. Blumenthal, *Theory and Applications of Distance Geometry*, Chelsea Publ. Co., Bronx, New York, 1970.

[4] J.-B. Bost and A. Connes, *Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory*, Selecta Mathematica (N.S.) **1** (1995), 411–457.

[5] R. B. Brook, *A construction of the greatest ambit*, Mathematical Systems Theory **4** (1970), 243–248.

[6] R. Cabello Sánchez, *Regards sur le problème des rotations de Mazur*, in *II Congress on Examples and Counterexamples in Banach Spaces (Badajoz, 1996)*, Extracta Mathematica **12** (1997), 97–116.

[7] R. Ellis, *Universal minimal sets*, Proceedings of the American Mathematical Society **11** (1960), 540–543.

[8] R. Exel and T. A. Loring, *Finite-dimensional representations of free product C^*-algebras*, International Journal of Mathematics **3** (1992), 469–476.

[9] J. M. G. Fell, *Weak containment and induced representations of groups*, Canadian Journal of Mathematics **14** (1962), 237–268.

[10] S. Gao and A. S. Kechris, *On the classification of Polish metric spaces up to isometry*, Caltech preprint, March 2000, 88 pp.

[11] T. Giordano and V. Pestov, a note in preparation.

[12] S. Glasner, *On minimal actions of Polish groups*, Topology and its Applications **85** (1998), 119–125.

[13] K. M. Goodearl and P. Menal, *Free and residually finite dimensional C^*-algebras*, Journal of Functional Analysis **90** (1990), 391–410.

[14] W. H. Gottscalk and G. A. Hedlund, *Topological Dynamics*, Vol. 36, American Mathematical Society Publications, Providence, 1955.

[15] M. I. Graev, *Free topological groups*, American Mathematical Society Translations **35** (1951), 61 pp.

[16] R. L. Graham, *Recent trends in Euclidean Ramsey theory*, Discrete Mathematics **136** (1994), 119–127.

[17] E. Granirer, *Extremely amenable semigroups 2*, Mathematica Scandinavica **20** (1967), 93-113.

[18] M. Gromov, *Filling Riemannian manifolds,* Journal of Differential Geometry **18** (1983), 1–147.

[19] M. Gromov, *Metric Structures for Riemannian and Non-Riemannian Spaces,* Progress in Mathematics **152**, Birkhäuser Verlag, Boston, 1999.

[20] M. Gromov and V. D. Milman, *A topological application of the isoperimetric inequality,* American Journal of Mathematics **105** (1983), 843–854.

[21] M. Hall, jr., *The Theory of Groups,* AMS Chelsea Publ., American Mathematical Society, Providence, R.I., 1999 reprint of the 1959 original.

[22] P. de la Harpe, *Moyennabilité de quelques groupes topologiques de dimension infinie,* Comptes Rendus de l'Académie des Sciences, Paris, Série A **277** (1973), 1037–1040.

[23] P. de la Harpe, *Moyennabilité du groupe unitaire et propriété P de Schwartz des algèbres de von Neumann,* in *Algèbres d'opérateurs (Sém., Les Plains-sur-Bex, 1978)*, Lecture Notes in Mathematics **725**, Springer-Verlag, Berlin, 1979, pp. 220–227.

[24] P. de la Harpe and A. Valette, *La propriété* (T) *de Kazhdan pour les groupes localements compacts,* Astérisque **175** (1989), 158 pp.

[25] S. Hartman and J. Mycielski, *On the imbedding of topological groups into connected topological groups,* Colloquium Mathematicum **5** (1958), 167–169.

[26] W. Herer and J. P. R. Christensen, *On the existence of pathological submeasures and the construction of exotic topological groups,* Mathematische Annalen **213** (1975), 203–210.

[27] M. Katětov, *On universal metric spaces,* in *General Topology and its Relations to Modern Analysis and Algebra VI,* Proceedings of the Sixth Prague Topology Symposium 1986 (Z. Frolík, ed.), Heldermann Verlag, Berlin, 1988, pp. 323–330.

[28] J. Keesling, *Topological groups whose underlying spaces are separable Fréchet manifolds,* Pacific Journal of Mathematics **44** (1973), 181–189.

[29] P. Komjath, *Ramsey-type results for metric spaces,* Journal of Combinatorial Theory, Series A **45** (1987), 323–328.

[30] J. Matoušek, *Ramsey-like properties for bi-Lipschitz mappings of finite metric spaces,* Commentationes Mathematicae Universitatis Carolinae **33** (1992), 451–463.

[31] V. D. Milman, *A new proof of the theorem of A. Dvoretzky on sections of convex bodies,* Functional Analysis and its Applications **5** (1971), 28–37.

[32] V. D. Milman, *Diameter of a minimal invariant subset of equivariant Lipschitz actions on compact subsets of* $\mathbb{R}^n$, in *Geometrical Aspects of Functional Analysis (Israel Seminar, 1985–86)*, Lecture Notes in Mathematics **1267**, Springer-Verlag, Berlin, 1987, pp. 13–20.

[33] V. D. Milman, *The heritage of P. Lévy in geometric functional analysis*, Astérisque **157–158** (1988), 273–301.

[34] V. D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces*, Lecture Notes in Mathematics **1200**, Springer-Verlag, Berlin, 1986.

[35] A. T. Paterson, *Amenability*, Mathematical Surveys and Monographs **29**, American Mathematical Society, Providence, RI, 1988.

[36] A. L. T. Paterson, *Nuclear C^*-algebras have amenable unitary groups*, Proceedings of the American Mathematical Society **114** (1992), 719–721.

[37] V. G. Pestov, *On free actions, minimal flows, and a problem by Ellis*, Transactions of the American Mathematical Society **350** (1998), 4149–4165.

[38] V. Pestov, *Some universal constructions in abstract topological dynamics*, in *Topological Dynamics and its Applications. A Volume in Honor of Robert Ellis*, Contemporary Mathematics **215** (1998), 83–99.

[39] V. G. Pestov, *Amenable representations and dynamics of the unit sphere in an infinite-dimensional Hilbert space*, Geometric and Functional Analysis **10** (2000), 1171–1201.

[40] G. Schechtman, *Lévy type inequality for a class of finite metric spaces*, in *Martingale Theory in Harmonic Analysis and Banach Spaces (Cleveland, Ohio, 1981)*, Lecture Notes in Mathematics **939**, Springer, Berlin–New York, 1982, pp. 211–215.

[41] M. Talagrand, *Concentration of measure and isoperimetric inequalities in product spaces*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **81** (1995), 73–205.

[42] S. Teleman, *Sur la représentation linéaire des groupes topologiques*, Annales Scientifiques de l'École Normale Supérieure **74** (1957), 319–339.

[43] P. Urysohn, *Sur un espace métrique universel*, Comptes Rendus de l'Académie des Sciences, Paris **180** (1925), 803–806.

[44] P. Urysohn, *Sur un espace métrique universel*, Bulletin des Sciences Mathématiques **51** (1927), 43–64, 74–90.

[45] V. V. Uspenskiĭ, *On the group of isometries of the Urysohn universal metric space*, Commentationes Mathematicae Universitatis Carolinae **31** (1990), 181-182.

[46] V. V. Uspenskiĭ, *Free topological groups of metrizable spaces*, Mathematics of the USSR-Izvestiya **37** (1991), 657–680.

[47] V. V. Uspenskij, *On subgroups of minimal topological groups*, Ohio University 1998 preprint, e-print available at http://arXiv.org/abs/math.GN/0004119.

[48] W. A. Veech, *Topological dynamics*, Bulletin of the American Mathematical Society **83** (1977), 775–830.

[49] A. N. Vershik, *Amenability and approximation of infinite groups*, Selecta Mathematica Soviet **2** (1982), 311–330.

[50] A. M. Vershik, *The universal Urysohn space, Gromov metric triples and random metrics on the natural numbers*, Russian Mathematical Surveys **53** (1998), 921–928.

[51] J. de Vries, *Topological Transformation Groups 1: A Categorical Approach*, Mathematical Centre Tracts **65**, Amsterdam, 1975.